Section 3.1 – Extreme Values

Example 1: Given the following is the graph of \( f(x) \)

Where is the maximum (x-value)?

What is the maximum (y-value)?

Where is the minimum (x-value)?

What is the minimum (y-value)?

1. Definition. Let \( c \) be in the domain of \( f \). Then \( f(c) \) is the
   - Absolute maximum if \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \).
   - Absolute minimum if \( f(c) \leq f(x) \) for all \( x \) in the domain of \( f \).
   The maximum and minimum values are also called the **extreme values**.

2. Definition. \( f(c) \) is a
   - Local maximum if \( f(c) \geq f(x) \) for all \( x \) near \( c \).
   - Local minimum if \( f(c) \leq f(x) \) for all \( x \) near \( c \).

(We’ll skip the formal definition of near \( c \))
Note that we do not have an absolute maximum at $x = 8$.

3. The **extreme value theorem**. If $f$ is **continuous** on a **closed interval**, then $f$ has an **absolute** maximum and an **absolute** minimum at some points in that closed interval.

Formally: If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

Note that in this example, the absolute maximum occurs at two different values of $x$. 
If either the continuity or closed interval hypothesis are ignored then a function does necessarily have extreme values

This function is not continuous, and while \( f \) has an absolute minimum \( f(2) = 0 \), it does not have an absolute maximum

This function is not defined on a closed interval and has no extreme values

4. Fermat’s Theorem. If \( f \) has a local maximum or minimum at \( c \) and if \( f'(c) \) exists, then \( f'(c) = 0 \).

But be careful. Just \( f'(c) = 0 \) will not guarantee that \( f(c) \) is a local min or max.

Example 2: \( f(x) = x^3 \)

\[ f'(x) = 3x^2 \]

and

\[ f'(0) = 0 \]

but

\[ f(0) \] is not a local minimum or maximum
Example 3: \( f(x) = |x| \) Looking at the graph we can see that \( f \) has a local and absolute minimum at \( f(0) = 0 \), but we have seen in previous examples that \( f'(0) \) is not defined (since the left and right hand derivatives do not match)

5. A critical number of a function of a function \( f \) is a number \( c \) in the domain of \( f \) such that either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

6. If \( f \) has a local maximum or minimum at \( c \), then \( c \) is a critical number of \( f \).

Now all this together leads us to…

7. The Closed Interval Method to find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\):
   1. Find the values of \( f \) at the critical number of \( f \) in \((a, b)\)
   2. Find the values of \( f \) at the endpoints of the interval
   3. The largest value is the absolute maximum and the smallest value is the absolute minimum

Example 4: Consider the function \( f(x) = 5 - 3x^2 \quad -1 \leq x \leq 4 \)

Find the critical values of \( f(x) \)

Compute steps 1 and 2 in the closed interval method

The absolute maximum is ___________ and this occurs at \( x = \) ___________

The absolute minimum is ___________ and this occurs at \( x = \) ___________
Example 5: Consider the function \( f(z) = \frac{1}{z-2} \) on the interval \([-4, -2]\)

Find the critical values of \( f(z) \)

Compute steps 1 and 2 in the closed interval method

The absolute maximum is \_____________ and this occurs at \( z = \_____________ \)

The absolute minimum is \_____________ and this occurs at \( z = \_____________ \)

Example 6: Let \( f(t) = \sin t \) on the interval \( \left[ -\frac{5\pi}{6}, \frac{\pi}{6} \right] \)

Find the critical values of \( f(t) \)

Compute steps 1 and 2 in the closed interval method

The absolute maximum is \_____________ and this occurs at \( t = \_____________ \)

The absolute minimum is \_____________ and this occurs at \( t = \_____________ \)
Example 7: Let $f(x) = 4 - |x|$ on the interval $[-9, 6]$

Find the critical values of $f(x)$

Compute steps 1 and 2 in the closed interval method

The absolute maximum is ____________ and this occurs at $x = ____________$

The absolute minimum is ____________ and this occurs at $x = ____________$
Section 3.2 – Mean Value Theorem

<table>
<thead>
<tr>
<th>Reminders</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Continuous</strong></td>
</tr>
<tr>
<td>A function ( f ) is <strong>continuous at a number</strong> ( a ) if ( \lim_{x \to a} f(x) = f(a) )</td>
</tr>
<tr>
<td>A function ( f ) is <strong>continuous on an interval</strong> if it is continuous at every number in that interval.</td>
</tr>
<tr>
<td><strong>Think</strong>: Continuous means you can draw the graph without lifting your pencil</td>
</tr>
</tbody>
</table>

| **Differentiable** |
| A function \( f \) is **differentiable at** \( a \) if \( f'(a) \) exists. It is **differentiable on an open interval** \((a, b)\) if it is differentiable at every number in the interval. |
| **Think**: Differentiable means that the graph is smooth (no pointy corners) and continuous |

**The Mean Value Theorem** (MVT)

Let \( f \) be a function that satisfies the following hypotheses:

1. \( f \) is **continuous** on the closed interval \([a, b]\)
2. \( f \) is **differentiable** on the open interval \((a, b)\)

Then there is a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

**Example 1**: Show that the function \( f(x) = 7 - 2x^2 \) satisfies the Mean Value Theorem for \( x \) in the interval \([-1, 4]\)
Example 2: Show that the function $f(x) = 2x^3 - 6x$ satisfies the MVT for $x$ in $[-2, 2]$.

Rolle’s Theorem
Let $f$ be a function that satisfies the following hypotheses:

1. $f$ is **continuous** on the closed interval $[a, b]$   
2. $f$ is **differentiable** on the open interval $(a, b)$  
3. $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.

Example 3: Show that $f(x) = x^2 - 4x + 4$ satisfies Rolle’s Theorem on $[0, 4]$. 

Several tangent lines are drawn on the graph of this function. Notice that all of the lines between points A and B and between C and D have positive slope and the graph is increasing. Also, the lines between points B and C have negative slopes and the graph is decreasing.

### Increasing/Decreasing Test

1. If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval
2. If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on that interval

### Example 1: Find where \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \) is increasing and decreasing

We need to find where the derivative of \( f \) is positive and negative, which means we need to find the critical numbers of \( f \).

\[
f'(x) = 12x^3 - 12x^2 - 24x
= 12x(x^2 - x - 2)
= 12x(x - 2)(x + 1)
\]

So our critical numbers are \(-1, 0, 2\). Next, the sign of the derivative depends on the sign of the three factors \(12x, (x - 2), (x + 1)\)

<table>
<thead>
<tr>
<th>Interval</th>
<th>(12x)</th>
<th>(x - 2)</th>
<th>(x + 1)</th>
<th>(f'(x))</th>
<th>(f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x &lt; -1)</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>decreasing on ((-\infty, -1))</td>
</tr>
<tr>
<td>(-1 &lt; x &lt; 0)</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>+</td>
<td>increasing on ((-1, 0))</td>
</tr>
<tr>
<td>(0 &lt; x &lt; 2)</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>decreasing on ((0, 2))</td>
</tr>
<tr>
<td>(x &gt; 2)</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>increasing on ((2, \infty))</td>
</tr>
</tbody>
</table>
The graph of \( f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \) confirms what we found in our table above.

Looking back at the graph in Example 1, \( f \) has a local max at \( f(0) \) and local minimums at \( f(-1) \) and \( f(2) \). Looking back at the signs of the derivative in the table leads to the first derivative test.

**The First Derivative Test**

1. If \( f' \) changes from **positive** to negative at \( c \), then \( f \) has a local **maximum** at \( c \).
2. If \( f' \) changes from **negative** to positive at \( c \), then \( f \) has a local **minimum** at \( c \).
3. If \( f' \) does not change sign at \( c \), then \( f \) does not have a local max or min at \( c \).
Example 2: Find all the local minimums and maximums of \( g(x) = x + 2 \sin x \) for \( 0 \leq x \leq 2\pi \)

Since \( g'(x) = 1 + 2 \cos x \), we need to solve \( \cos x = -1/2 \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>( g'(x) = 1 + 2 \cos x )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; x &lt; \frac{2\pi}{3} )</td>
<td>+</td>
<td>increasing on ( (0, \frac{2\pi}{3}) )</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} &lt; x &lt; \frac{4\pi}{3} )</td>
<td>-</td>
<td>decreasing on ( \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) )</td>
</tr>
<tr>
<td>( \frac{4\pi}{3} &lt; x &lt; 2\pi )</td>
<td>+</td>
<td>increasing on ( (\frac{4\pi}{3}, 2\pi) )</td>
</tr>
</tbody>
</table>

Using the first derivative test, there is a local maximum at

\[
g \left( \frac{2\pi}{3} \right) = \frac{2\pi}{3} + 2 \sin \frac{2\pi}{3} = \frac{2\pi}{3} + \sqrt{3} \approx 3.83
\]

And a local minimum at

\[
g \left( \frac{4\pi}{3} \right) = \frac{4\pi}{3} + 2 \sin \frac{4\pi}{3} = \frac{4\pi}{3} - \sqrt{3} \approx 2.46
\]

Example 3: Find the locations (x-values) of the local extrema of the following functions

\[
f(x) = 53x^4 - 42 \quad g(x) = 3x + \frac{12}{x} \quad h(x) = 5(x - 7)^{2/3} + 6
\]
Concavity
If a graph lies above all of its tangents, then it is **concave up**. If a graph lies below all of its tangents, then it is **concave down**.

Just look at the direction that the graph is curving. Opening up is concave up, opening down is concave down.

Concavity Test
1. If $f''(x) > 0$ for all $x$ in an interval, then the graph of $f$ is concave up on that interval
2. If $f''(x) < 0$ for all $x$ in an interval, then the graph of $f$ is concave down on that interval

Inflection Points
A point $P$ on a curve $y = f(x)$ is an **inflection point** if $f$ is continuous at $P$ and the concavity changes from up to down (or down to up) at $P$. 
Example 4: Given that $f(x) = 5x^3 + 2x - 17$, find all intervals on which $f$ is concave up and concave down. What are the inflection points of $f$ (if any)?

Example 5: Find the local minimum of $f(x) = x^2 - 3$. What is the concavity of $f$?

Example 6: Find the local maximum of $f(x) = 10 + 4x - x^2$. What is the concavity of $f$?
The Second Derivative Test

Suppose $f''$ is continuous near $c$.

1. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.

2. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

Example 7: Discuss $f(x) = x^4 - 8x^3$ with respect to concavity, points of inflection, local minimums and maximums, and sketch the graph.

Note: The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 7).
Section 3.4 – Limits at Infinity

1. Let $f$ be a function defined on some interval $(a, \infty)$. Then
   \[ \lim_{x \to \infty} f(x) = L \]
   means that $f(x)$ gets arbitrarily close to $L$ as $x$ approaches $\infty$.

2. Let $f$ be a function defined on some interval $(-\infty, a)$. Then
   \[ \lim_{x \to -\infty} f(x) = L \]
   means that $f(x)$ gets arbitrarily close to $L$ as $x$ approaches $-\infty$.

3. The line $y = L$ is called a horizontal asymptote if either
   \[ \lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L \]

Some graphs have two horizontal asymptotes.
Example 1: Find the following limits

\[ \lim_{x \to \infty} \frac{1}{x} \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x} \]

4. If \( r > 0 \) is a rational number, then

\[ \lim_{x \to \infty} \frac{1}{x^r} = 0 \]

If \( r > 0 \) is a rational number such that \( x^r \) is defined for all \( x \), then

\[ \lim_{x \to -\infty} \frac{1}{x^r} = 0 \]

Example 2: Find the following limit

\[ \lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \]

Example 3: Find the following limit

\[ \lim_{x \to \infty} \frac{8 + 7\sqrt[3]{x}}{8 - 7\sqrt[3]{x}} \]
Example 4: Find the following limit

$$\lim_{x \to \infty} \frac{6 + 2x^{4/3} - x}{9x^{7/5} - 2}$$

Example 5: Find the following limit

$$\lim_{x \to \infty} \frac{(3 - x)(7 + 5x)}{(1 - 7x)(8 + 6x)}$$

Example 6: Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$
Example 7: Find the following limit

\[
\lim_{x \to \infty} \left( \sqrt{x^2 + 1} - x \right)
\]

Example 8: Find the following limit

\[
\lim_{x \to \infty} \left( \sqrt{49x^2 + 53x - 7x} \right)
\]
Infinite Limits at Infinity

Example 9: Find the limits \( \lim_{x \to \infty} x^3 \) and \( \lim_{x \to -\infty} x^3 \)

Example 10: Find \( \lim_{x \to \infty} (x^2 - x) \)

<table>
<thead>
<tr>
<th>DO NOT SOLVE THE PROBLEM WITH THIS METHOD</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \infty - \infty )</td>
</tr>
</tbody>
</table>

| THIS SOLUTION IS WRONG |

Instead, use limit laws correctly to get

\[ \lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty \]

Example 11: Find the following limit

\[ \lim_{x \to \infty} \frac{x^2 + x}{3 - x} \]
Section 3.5 – Curve Sketching

Guidelines for Sketching a Curve
A. Domain
B. Intercepts
C. Symmetry (or period)
D. Asymptotes
E. Increasing/Decreasing
F. Local Min and Max
G. Concavity

Example 1: Sketch the graph of

\[ f(x) = \frac{2x^2}{x^2 - 1} \]
Example 2: Sketch the graph of

\[ f(x) = \frac{x^2}{\sqrt{x + 1}} \]
Example 3: Sketch the graph of

\[ f(x) = \frac{\cos x}{2 + \sin x} \]
Slant (Oblique) Asymptotes

A curve has a **slant asymptote** (instead of a horizontal asymptote) if

\[
\lim_{x \to \infty} [f(x) - (mx + b)] = 0
\]

For rational functions, there is a slant asymptote whenever the degree of the numerator is exactly one larger than the degree of the denominator. The equation of the slant asymptote can be attained by polynomial long division.

**Example 4:** Find all of the asymptotes of

\[
f(x) = \frac{x^3}{x^2 + 1}
\]
Example 5: Sketch the graph of

\[ f(x) = \frac{x^3}{x^2 + 1} \]
Example 6: Sketch the graph of

\[ f(x) = \frac{x^3 - 4}{x^2} \]
Example 7: Sketch the graph of

\[ f(x) = 4x^{1/3} + x^{4/3} \]
Example 8: Sketch the graph of

\[ f(x) = \frac{x^2 + x - 1}{x - 1} \]
Section 3.7 – Optimization

Example 1: A farmer has 2400 feet of fencing and wants fence off a rectangular field along a river. The edge on the river needs no fence. What is the maximum area he can inclose?

Algebra way:

Calculus way:
Recall

The Closed Interval Method to find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\):

1. Find the values of \( f \) at the critical number of \( f \) in \((a, b)\)
2. Find the values of \( f \) at the endpoints of the interval
3. The largest value is the absolute maximum and the smallest value is the absolute minimum

Optimization Methods:

1. **Understand the Problem** – What are you trying to find? What are you given? What previous information do you know (formulas, etc.)?
2. **Draw a Diagram** – Most of the time it is helpful to draw a picture that visualizes the problem.
3. **Introduce Notation** – Pick a symbol for the quantity you are trying to optimize (I’ll pick \( Q \) for now). Assign symbols \((a, b, c, ... x, y, etc.)\) to any other unknowns. It’s usually helpful to pick symbols that relate to that quantity. \( A \) for area, \( h \) for height, \( t \) for time.
4. Express \( Q \) in terms of the symbols from step 3.
5. If \( Q \) is expressed as a function with more than one variable, use other formulas or given information to make substitutions until \( Q \) is expressed as a function of just one variable, \( Q = f(x) \). Find the domain of this function.
6. Use the maximum and minimum tests we have learned to find the absolute minimum or maximum value of \( f \).

Example 2: You are trying to make a steel, cylindrical can that holds 1 L (1000 cm\(^3\)) of oil. If your goal is to use the least amount of steel necessary, what should the dimensions of the can be?
Example 3: A rectangle is drawn inside of the parabola \( y = 3 - x^2 \). What is the maximum area that can be enclosed?

Example 4: A rancher wants to fence in a rectangular area of 726 square meters in a field and then divide the region in half with a fence down the middle parallel to one side. What is the smallest length of fencing that will be required to do this?
**Example 5:** If a total of 300 square centimeters of material is to be used to make a box with a square base and an open top, find the largest possible volume of such a box.

Example 6: A store has been selling 200 Quibles a week at $350 each. Some study says that for each $10 rebate offered, the number of Quibles sold will increase by 20 per week. Find the demand and revenue functions. How large a rebate should the store offer to maximize revenue?
**Section 3.8 – Newton Method**

Suppose that a car dealer offers to sell you a car for $18,000 or for payments of $375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation

\[ 48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0 \]

To get an *approximate* solution, we can plot the left side of the equation in a graphing calculator.

![Graph showing the left side of the equation](image)

If we zoom in we can see that the root is about 0.0076. To get an even more accurate answer we can use the calculators built in root finder and get 0.007628603.

So, how did the calculator get that answer?

---

**Newton’s Method**

Start with the point \((x_1, f(x_1))\) and find the tangent line. Call the x-intercept of this line \((x_2, 0)\). The line between them is

\[ 0 - f(x_1) = f'(x_1)(x_2 - x_1) \]

which can be solved for \(x_2\)

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]

If we start the same process over again,

\[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \]

Or in general,

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

If these numbers get closer to the root \(r\) as \(n\) becomes large, then the sequence \(x_n\) converges to \(r\)

\[ \lim_{n \to \infty} x_n = r \]
Example 1: Starting with $x_1 = 2$, find the third approximation $x_3$ to the root of the equation $x^3 - 2x - 5 = 0$.

Apply Newton’s method with $f(x) = x^3 - 2x - 5$ and $f'(x) = 3x^2 - 2$. So

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$

$$x_2 = 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1$$

With $n = 2$

$$x_3 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0941$$

Example 2: Suppose we want to use Newton’s Method to approximate $\sqrt{53}$. Select a function $f(x)$ that has will have $\sqrt{53}$ as a root and pick a good choice for $x_1$

Example 3: Suppose we want to use Newton’s Method to approximate $\frac{3}{\sqrt{25}}$. Select a function $f(x)$ that has will have $\frac{3}{\sqrt{25}}$ as a root and pick a good choice for $x_1$