Section 2.1 – Derivatives

1. The **tangent line** to the curve $y = f(x)$ through the point $P(a, f(a))$ is the line through point $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Provided that this limit exists.

**Example 1:** Find the equation of the tangent line to the parabola $f(x) = x^2$ at the point $P(1,1)$

2. Revisit 1, but make the substitutions $h = x - a$ and $x = a + h$. Note that as $x$ approaches $a$, $h$ approaches 0. We get the new equation

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Which should remind you of the difference quotient.

**Example 2:** Find the slope from Example 1 again using this new formula.
3. Recall that the average rate of change of \( f \) between points \( x_1 = a \) and \( x_2 = a + h \) is

\[
\text{average velocity} = \frac{f(a + h) - f(a)}{a + h - a}
\]

If we want to get the **instantaneous velocity** of \( f \) at \( a \), then we want the two points we pick to be as close as possible. We can do this by taking the limit as \( h \) goes to 0

\[
v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

**Example 3:** From the basic kinematic equations in physics, the position of an object undergoing constant acceleration at time \( t \) is given by the equation \( x(t) = x_0 + v_0 t + \frac{1}{2} at^2 \) where \( x_0 \) is the initial position, \( v_0 \) is the initial velocity, and \( a \) is the acceleration of the object. Find the instantaneous velocity at time \( t = t_0 \) of a ball dropped from a height of 10m. Use \( a = -9.8 \text{m/s}^2 \) for the acceleration due to gravity.

Now find the instantaneous velocity again if the ball was thrown upward with an initial velocity of \( 5 \text{m/s} \).

Another basic kinematic equation is \( v(t) = v_0 + at \). Notice any similarities to our answer?
4. In mathematics, the instantaneous rate of change of a function has a name, called the derivative. The derivative of a function $f$ at a number $a$ denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if the limit exists.

5. Going back to the idea of tangent lines, we can also write the derivative as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

**Example 4:** Find the derivative of $f(x) = 3x^2 - x + 2$ when $x = -2$.

**Example 5:** Find $f'(7)$ if

$$f(x) = \frac{1}{x - 4}$$
Example 6: Find $f'(16)$ if $f(x) = 5\sqrt{x} - 13$

Example 7: Find $f'(0)$ if

$$f(x) = \begin{cases} 
-5x^2 + 3x & \text{if } x < 0 \\
4x^2 + 3x & \text{if } x \geq 0 
\end{cases}$$

Example 8: Find $f'(0)$ if

$$f(x) = \begin{cases} 
-5x^2 + 3x & \text{if } x < 0 \\
4x^2 - 3x & \text{if } x \geq 0 
\end{cases}$$
Section 2.2 – Derivative Function

1. Recall the definition of a derivative at a point:
   \[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \]

2. The formal definition of the derivative as a function is very much the same. The difference being that instead of taking a fixed point \( a \), we use a variable \( x \).
   \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
Example 1: Using the definition of a derivative, find the derivative of \( f(x) = x^3 - x \)

Example 2: Using the definition of a derivative, find the derivative of \( f(x) = \sqrt{x} \)

Example 3: Find \( f'(x) \) using the definition of a derivative given that
\[
f(x) = \frac{7 - x}{3 + x}
\]

Other Notations: Derivatives have several representations that are commonly used. Here are a few
\[
f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)
\]

3. A function \( f \) is **differentiable at \( a \)** if \( f'(a) \) exists. It is **differentiable on an open interval** (\( a, b \)) if it is differentiable at every number in the interval.
Example 4: Is \( f(x) = |x| \) differentiable?

4. **Theorem**: If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

This also means that if \( f \) is **not** continuous at \( a \), then \( f \) is **not** differentiable at \( a \).

Note that the converse (if \( f \) is continuous at \( a \), then \( f \) is differentiable at \( a \)) is not necessarily true. Just look at Example 4.

**How can a function fail to be differentiable?**

(c) A vertical tangent

(b) A discontinuity
Higher Derivatives

5. **Second Derivative:** As discussed earlier, if \( f \) is differentiable then its derivative is a function. So \( f' \) may have a derivative of its own called the **second derivative**, denoted \( (f')' = f'' \).

\[
\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2}
\]

The second derivative is the slope of the derivative. This translates to the **concavity** of the original function \( f \).

6. **Third Derivative:** The third derivative \( f''' \) is the derivative of the second derivative \( (f'')' = f''' \)

\[
y''' = f'''(x) = \frac{d}{dx} \left( \frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3}
\]

This process can be continued as long as we still have a differentiable function, but we tend to stop using apostrophes after the third derivative. The fourth derivative is \( f^{(4)} \).

\[
y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}
\]

**Example 5:** If \( f(x) = x^3 - x \), find \( f''(x), f'''(x), \) and \( f^{(4)}(x) \)
The graphs of function and first two derivatives from Example 5.
Section 2.3 – Derivative Formulas

Example 1: (Slopes) What is the derivative of $f(x) = 6$?

What is the derivative of $f(x) = 3x - 7$?

What is the derivative of $y = mx + b$?

Example 2: Using the formal definition of a derivative and binomial coefficients (Pascal’s Triangle), find $f'(x)$ if $f$ is the polynomial $f(x) = ax^n$

Derivative Rules

1. Derivative of a constant function:
   \[
   \frac{d}{dx}(c) = 0
   \]

2. Derivative of a linear function:
   \[
   \frac{d}{dx}(x) = 1
   \]

3. The Power Rule: If $n$ is a positive integer, then
   \[
   \frac{d}{dx}(x^n) = nx^{n-1}
   \]
   Note that this means we can use these rules in place of the formal definition of a derivative.
Example 3: Find the derivative of \( f(x) = 6x^3 \)

4. The **Constant Multiple Rule**: If \( c \) is a constant and \( f \) is a differentiable function, then

\[
\frac{d}{dx} (cf(x)) = c \frac{d}{dx} f(x)
\]

**Proof**: Let \( g(x) = cf(x) \). Then

Example 4: Find the derivative of \( y = x^4 + x^2 \)
5. The **Sum Rule**: If \( f \) and \( g \) are both differentiable, then

\[
\frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

**Proof**: Let \( F(x) = f(x) + g(x) \). Then

6. The **Difference Rule**: If \( f \) and \( g \) are both differentiable, then

\[
\frac{d}{dx} (f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)
\]

The proof is the same as the sum rule, but let \( F(x) = f(x) + (-g(x)) \)

**Example 5**: \( y = x^9 + 4x^4 - 14x^3 + 5x^2 + 53x - 117 \)

\[
\frac{dy}{dx} =
\]
Example 5: Let $f(x) = 5x^3$ and $g(x) = 2x^7$

$f'(x) = \hspace{2cm} g'(x) = \hspace{2cm} f'(x) \cdot g'(x) = \hspace{2cm}

(f(x) \cdot g(x))' = \hspace{2cm}

7. The Product Rule: If $f$ and $g$ are both differentiable, then

$$\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [f(x)]$$

Proof: Let $F(x) = f(x)g(x)$. Then

Example 6: Let $F(x) = (5x^3)(2x^7)$. Use the product rule to find $F'(x)$
Example 7: Let \( f(x) = 5x^3 \) and \( g(x) = 2x^7 \)
\[
\frac{g'(x)}{f'(x)} = \quad \left( \frac{g(x)}{f(x)} \right)' = \]

8. The Quotient Rule: If \( f \) and \( g \) are both differentiable, then
\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}
\]

Proof: Let \( F(x) = f(x)/g(x) \). Then

Remember: \textbf{GiraF} \textbf{fes ForGo}
Example 8: Find $f'(x)$ using the quotient rule given that

$$f(x) = \frac{7 - x}{3 + x}$$

Compare this result to Section 2.2 Example 3

Example 9: Use the quotient rule to find $y'$ if $y = x^{-3}$

<table>
<thead>
<tr>
<th>9.</th>
<th>If $n$ is a positive integer, then $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof:</td>
<td></td>
</tr>
</tbody>
</table>
Recall from Section 2.2 Example 2 we saw that

\[ \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \]

Which can be written as

\[ \frac{d}{dx} (x^{1/2}) = \frac{1}{2} x^{-1/2} \]

10. The **Power Rule** (General Version): If \( n \) is any real number, then

\[ \frac{d}{dx} (x^n) = n x^{n-1} \]

**Proof:** Google it (if that’s what you’re into)

**Example 10:** Use the general **power rule**:

a) \( f(x) = x^\pi \) \quad \Rightarrow \quad f'(x) =

b) \( y = \frac{1}{\sqrt{x^2}} \) \quad \Rightarrow \quad y' =

**Example 11:** Differentiate \( f(t) = \sqrt{t}(a + bt) \) with the:

<table>
<thead>
<tr>
<th>Product rule</th>
<th>Power rule</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 12: Find the equations of the tangent and normal lines to the curve \( y = \sqrt{x/(1 + x^2)} \) at the point \( \left(1, \frac{1}{2}\right) \).

All the rules in one table

\[
\begin{align*}
\frac{d}{dx}(c) &= 0 & \frac{d}{dx}(x^n) &= nx^{n-1} \\
(cf)' &= cf' & (f + g)' &= f' + g' \\
(fg)' &= fg' + gf' & \left(\frac{f}{g}\right)' &= \frac{gf' - fg'}{g^2}
\end{align*}
\]
Section 2.4 – Trig Derivatives

Quick trig lesson:

Arc length \( s = r\theta \)

Angle sum formula: \( \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \)

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \]

Other Helpful identities:

\[ \sin^2 x + \cos^2 x = 1 \quad \sin 2x = 2 \sin x \cos x \quad \cos 2x = \cos^2 x - \sin^2 x \]

Finding the derivative of \( f(x) = \sin x \):

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \\
= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
= \lim_{h \to 0} \left[ \sin x \cos h - \sin x \frac{h}{h} + \cos x \sin h \frac{h}{h} \right] \\
= \lim_{h \to 0} \left[ \sin x \left( \frac{\cos h - 1}{h} \right) + \cos x \left( \frac{\sin h}{h} \right) \right] \\
= \sin x \cdot \lim_{h \to 0} \left( \frac{\cos h - 1}{h} \right) + \cos x \cdot \lim_{h \to 0} \left( \frac{\sin h}{h} \right)
\]

So what we need is to find two limits.

\[ \lim_{h \to 0} \left( \frac{\cos h - 1}{h} \right) \text{ and } \lim_{h \to 0} \left( \frac{\sin h}{h} \right) \]

Turns out all it takes is a little geometry.

\[
\lim_{h \to 0} \left( \frac{\sin h}{h} \right) = 1
\]
Proof: \( |BC| = |OB| \sin \theta = \sin \theta \)

\( |BC| < |AB| < \arcsin AB = \theta \)

\[ \sin \theta < \theta \quad \Rightarrow \quad \frac{\sin \theta}{\theta} < 1 \]

\[
\theta = \arcsin AB < |AE| + |EB| \\
< |AE| + |ED| \\
= |AD| = |OA| \tan \theta \\
= \tan \theta
\]

\[ \theta < \tan \theta = \frac{\sin \theta}{\cos \theta} \]

\[ \cos \theta < \frac{\sin \theta}{\theta} < 1 \]

Now use the squeeze theorem!

\[ 1 = \lim_{\theta \to 0} \cos \theta \leq \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \leq \lim_{\theta \to 0} 1 = 1 \]

So,

\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \]

Now, finish the derivative of \( \sin x \)
Find the derivative of \( \cos x \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}
\]

### Derivatives of Trigonometric Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( \csc x )</td>
<td>( -\csc x \cot x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( \sec x )</td>
<td>( \sec x \tan x )</td>
</tr>
<tr>
<td>( \tan x )</td>
<td>( \sec^2 x )</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>( -\csc^2 x )</td>
</tr>
</tbody>
</table>
Example 1: Evaluate the limit
\[ \lim_{x \to 0} \frac{\sin x}{2x} \]

Example 2: Evaluate the limit
\[ \lim_{x \to 0} \frac{\sin 2x}{x} \]

Example 3: Evaluate the limit
\[ \lim_{x \to 0} \frac{\sin(x^2 + x)}{2x} \]

Example 4: Evaluate the limit
\[ \lim_{x \to 0} \frac{\sin(53x)}{\sin(42x)} \]
Example 5: $f(x) = 13 \sin x - 5 \cos x$

$f'(x) =$

Example 6: $f(t) = 2 \tan t - 7 \cot t + \sec t - 4 \csc t + 3 \sin t - 17 \cos t - t^5$

$f'(t) =$

Notice that the slopes of the tangent lines have the same value as the graph of the derivative below.
Section 2.5 – The Chain Rule

Example 1: Find the derivative of $3(2x^7)^4$

1. The Chain Rule: If $f$ and $g$ are both differentiable, then the composite function defined by $F(x) = f(g(x))$ is differentiable and
   
   $$F'(x) = f'(g(x)) \cdot g'(x)$$

   So if $F(x) = \sin(x^2)$, then $F'(x) = \cos(x^2) \cdot (2x)$

Example 2: Find the derivative of $f(x) = (x^8 - 7)^{90}$

Example 3: Find the derivative of $f(x) = 3 \tan(x^4 - x + 17)$
Example 4: Find the derivative of \( f(x) = \sqrt{x} \cdot \csc x^3 \)

Example 5: Find the derivative of \( f(x) = 4 \sec(7 \tan x^2 + \cos 2x) \)

Example 6: Find the derivative of \( f(x) = (5x^2 - 11x^6)^{-4} + 7x^3)^8 \)
Example 7: Find the derivative of \( f(x) = (7x^5 - 4x^3 \cdot \sin x - 16x^2 + 3)^{-2} \)

Example 8: Find the derivative of

\[
f(x) = \sqrt{\cos \left( \frac{\tan x^2}{x^4 + (2x)^{-4}} \right) + 13x^7 \cdot \sin(-x^5 + 3x)}
\]
Section 2.7 – Rates of Change

Physics! If $s = f(t)$ is the position function of a particle moving a straight line then use the following notation to talk about other properties of the particle.

1. Average Velocity:
   \[ \frac{\Delta s}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} \]
   
2. (Instantaneous) Velocity:
   \[ \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = v(t) = s'(t) \]
   Velocity has units of \textit{distance/time}. (For example \(\text{ft/min, m/s, mph}\))

3. Acceleration:
   \[ a(t) = v'(t) = s''(t) \]
   Acceleration has units of \textit{distance/time}^2. (For example \(\text{ft/s}^2\) as “feet per second per second”)

Example 1: Answer the following questions for a particle moving a straight line. $t$ is measured in seconds and $s$ is measured in feet.

<table>
<thead>
<tr>
<th>Position</th>
<th>Velocity</th>
<th>Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s(t) = t^3 - 8t^2 + 3t$</td>
<td>$v(t) = s'(t)$</td>
<td>$a(t) = s''(t)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Units:</th>
<th>Units:</th>
<th>Units:</th>
</tr>
</thead>
</table>

a) Find the velocity and acceleration at time $t$

b) Find the average velocity over the interval $[2, 8]$

c) Find $v(7)$

d) What is the velocity after 2 seconds?

e) What is the acceleration after 5 seconds?

f) When is the particle at rest?
g) For $t \geq 0$, when is the particle moving in a positive direction?

Example 2: Answer the following questions for a particle moving a straight line. $t$ is measured in seconds and $s$ is measured in meters.

<table>
<thead>
<tr>
<th>Position</th>
<th>Velocity</th>
<th>Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s(t) = t^3 - 15t^2 + 63t$</td>
<td>$s'(t) = v(t) =$</td>
<td>$s''(t) = a(t) =$</td>
</tr>
<tr>
<td>Units:</td>
<td>Units:</td>
<td>Units:</td>
</tr>
</tbody>
</table>

a) Find the velocity and acceleration at time $t$

b) When is the particle at rest?

c) Find the acceleration each time the velocity is 0.

d) When is the acceleration of the particle 0?

e) Find the speed of the particle when the acceleration is 0.

f) Find the total distance traveled by the particle from $t = 0$ to $t = 4$

g) What is the position of the particle at time $t = 4$?
Graphs of Position, Velocity, and Acceleration.

<table>
<thead>
<tr>
<th>Position:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Graph increasing = positive velocity. Graph decreasing = negative velocity</td>
</tr>
<tr>
<td>2. Top of a peak, bottom of a valley, or flat line = zero velocity</td>
</tr>
<tr>
<td>3. Graph opens up = positive acceleration. Graph opens down = negative acceleration</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Velocity:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Positive value = moving forward. Negative value = moving backward.</td>
</tr>
<tr>
<td>2. Speed up = slope points away from horizontal axis</td>
</tr>
<tr>
<td>3. Slow down = slope points toward horizontal axis</td>
</tr>
<tr>
<td>4. Graph increasing = positive acceleration. Graph decreasing = negative acceleration</td>
</tr>
</tbody>
</table>

**Example 3:** The graph below shows the velocity $v(t)$ of a particle.

![Graph of velocity v(t)](image)

a) When does the particle move forward?

b) When does the particle move backward?
c) When does the particle speed up?

d) When does the particle slow down?

e) When is the particle’s acceleration positive?

f) When is the particle’s acceleration negative?

g) When is the particle’s acceleration zero?

h) When does the particle move at its greatest speed?

i) When does the particle stand still for more than an instant?
Example 4: Imagine that an event planner for DTE Energy Music Theatre were planning an event and his marketing department had produced the following equation to model the expected number of guests $g$ as a function of ticket price $p$.

$$g(p) = -5p^2 + 2000$$

a) Find a function $R$ for the revenue of ticket sales in terms of ticket price.

b) Find a function for the marginal revenue, and explain the meaning of $R'(25)$.

c) What price would you suggest that tickets be sold at?
Section 2.6 – Implicit Differentiation

Implicit functions
Recall that an explicit function is defined so that every element of the domain is mapped to exactly one element of the range (every \(x\) goes to exactly one \(y\)).

Definition of implicit: (adj) implied, though not plainly expressed.

So an implicit function is one that is implied, although not plainly expressed as a function of \(x\). Take \(x^2 + y^2 = 25\) for example. We know from our pre-calculus course that this equation represents a circle of radius 5 centered at the origin. But if we try to express \(y\) as a function of \(x\), we are unable to do so since \(y = \pm \sqrt{25 - x^2}\) gives two \(y\) values for any \(x \in (-5, 5)\). So \(x^2 + y^2 = 25\) is an implicit function.

Implicit differentiation
When taking the derivative of an implicit function, we treat \(y\) as a function of \(x\) (\(y = f(x)\)). So

\[
\frac{d}{dx}(y(x)) = y'(x) = \frac{dy}{dx}
\]

This also means that the chain rule applies to implicit functions such as

\[
\frac{d}{dx}y^2 = 2y \frac{dy}{dx} = 2yy'
\]

Example 1: Find \(dy/dx\) if \(x\) and \(y\) satisfy the implicit function \(x^2 + y^2 = 16\)
**Example 2:** Find $dy/dx$ if $x$ and $y$ satisfy the implicit function $x^2y = 4$

**Example 3:** Find $dy/dx$ if $x$ and $y$ satisfy the implicit function $xy + \pi \cdot \sin y = \pi$

**Example 4:** If $x$ and $y$ satisfy the implicit function $3xy - 3x^5 = -8$, find

$$y' =$$

$$y'' =$$
Example 5: If \( x \) and \( y \) satisfy the implicit function \( x^3 + y^3 = 6xy \), find

\[
y' =
\]

\[
y'' =
\]

Example 6: If \( x \) and \( y \) satisfy the implicit function \( (x^2 + xy)^2 + \sin \left( \frac{y}{x} \right) = 1 \), find

\[
y' =
\]

The equation of the tangent line at the point \((1, 0)\)
**Example 7:** If $x$ and $y$ satisfy the implicit function $x^4 + y^4 = 16$, find

\[ y' = \]

\[ y'' = \]

**Example 8:** Suppose that $(y(x))^2 + 11x = x^2y(x) + 22$ and $y(2) = 4$. Find

\[ y' = \]

The equation of the tangent line at the point $(2, 4)$
Section 2.8 – Related Rates

Steve’s secret 6 step method for solving related rate problems
1. Read the question
2. Write down the appropriate formula
3. Take the derivative of said formula
4. Plug in the known quantities and solve
5. ???
6. Profit

Example 1: Air is being pumped into a spherical balloon at a rate of $64\text{ in}^3/\text{s}$. How fast is the radius increasing when the diameter is $8\text{ in}$?

How fast is the surface area increasing?

A couple of things to point out.
1. When taking the derivative we must pay attention to the derivative variable (it’s usually time).
2. Any variable that can change must be treated as a function that has its own derivative
3. Don’t forget the chain rule!
4. Some problems may require more than one equation. I think it’s usually best to substitute before you take a derivative
Some useful formulas for related rate problems

<table>
<thead>
<tr>
<th>Volume of Sphere</th>
<th>Surface Area of a Sphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V = \frac{4}{3}\pi r^3$</td>
<td>$A = 4\pi r^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume of a Cone</th>
<th>Area of Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2h$</td>
<td>$A = \pi r^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pythagorean Theorem</th>
<th>Area of a Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^2 + b^2 = c^2$</td>
<td>$A = \frac{1}{2}bh$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume of a Cube</th>
<th>Volume of a Cylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V = s^3$</td>
<td>$V = Bh = \pi r^2h$</td>
</tr>
</tbody>
</table>

**Example 2:** If the radius of circle is increasing at a rate of 5 m/s ($dr/dt = 5$), how fast is the area of the circle increasing? (Find $dA/dt$)

How fast is the diameter increasing?

How fast is the circumference increasing?
Example 3: A stone is dropped in the middle of still lake and creates a ripple. If the radius of the ripple increases at a constant rate of 6 $ft/s$ how fast is the area of the ripple increasing when the radius is 5 feet?

How fast is the area increasing after 4 seconds?

How fast is the radius increasing after 4 seconds?

Example 4: A spherical snowball is melting in a way such that its diameter decreases at a rate of 4 cm/min. How fast is the volume of the snowball decreasing when the diameter is 6 cm?
Example 5: A 13 ft ladder was leaning against a wall. Now its slipping down the wall at a constant rate of 7 ft/s. How fast is the bottom slipping out when it is 12 feet from the wall?

Example 6: A two-piece extension ladder leaning against a wall is collapsing (sliding in) at the rate of 2 feet per second at the same time as its foot is moving away from the wall at the rate of 3 feet per second. How fast is the angle the ladder makes with the ground changing when the ladder is 8 feet from the ground and the foot is 6 feet from the wall?
Example 7: A rectangular aquarium with a 50 cm by 80 cm base is being filled with water at a rate of 5000 cc/min. At what rate is the height of the water changing when the height is 15 cm?

Example 8: One car leaves Crunchy’s in East Lansing and travels due West going 48 km/h. Another car leaves Coral Gables in East Lansing and travels due South going 32 km/h. Coral Gables is 3 km due east of Crunchy’s. At what rate is the distance between the cars changing at the instant when the cars have traveled for 15 minutes?
Example 9: A water reservoir is shaped like a point down cone. The reservoir is 16 ft tall and the diameter at the top is 24 ft long. If the tank is being filled at a constant rate of 10 cubic feet per minute, how fast is the height increasing when the water is 8 feet deep?

Example 10: The height of a cone is decreasing at 3 cm/s while the radius is increasing at 2 cm/s. When the diameter is 8 cm and height is 6 cm, at what rate is the volume changing?
Section 2.9 – Linearization

The **linearization** of \( f \) at \( a \) is given by the formula

\[
L(x) = f(a) + f'(a)(x - a)
\]

and is used to approximate the function near the point \( a \).

\[
f(x) \approx f(a) + f'(a)(x - a)
\]

This is called the **linear approximation** of \( f \) at \( a \). We say that the linearization is centered at \( a \).

But really, this is just the **tangent line**

\[
y = y_1 + m(x - x_1)
\]

Since \( x_1 = a \), \( y_1 = f(a) \) and \( m = f'(x_1) = f'(a) \)

**Example 1**: Find the linearization \( L(x) \) of the function \( f(x) = 3x^2 + 7x \) centered at \( x = 3 \).

Use the linearization to approximate

\[
f\left(\frac{28}{10}\right) \approx
\]

**Example 2**: Use a linear approximation centered at 0 to estimate \( \cos(0.001) \)
Example 3: Use a linear approximation centered at 0 to estimate $\sin(0.01)$

Example 4: Use a linear approximation to estimate $\sin(34^\circ)$

Example 5: Use a linear approximation to estimate the amount of paint needed to apply a coat of paint $\frac{2}{15}$ cm thick to the surface of a cylindrical water tank that has a radius of 8 m and a height of 3 m. You do not need to paint the circular top or bottom