Some more basic facts about algebra: You should know, or be able to easily derive, the following facts:

\[(a + b)^2 = a^2 + 2ab + b^2\]

\[(a - b)^2 = a^2 - 2ab + b^2\]

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

\[(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\]

Of course, you need only remember the first and third of these formulas. The second and fourth follow by replacing \(b\) with \(-b\).

You should also know:

\[(a + b)(c + d) = ac + bc + ad + bd\]

\[(a + b)(a - b) = a^2 - b^2\]

and that (In the following, we assume \(a \neq 0, b \neq 0\), and \(n\) is an integer \(\geq 1\). We also make the customary assumption that any denominator is not zero, both in the statements below and in the exercises.)

(i) \(a^n - b^n\) is always divisible by \(a - b\). For example,

\[(a^2 - b^2)/(a - b) = a + b\]

\[(a^5 - b^5)/(a - b) = a^4 + a^3b + a^2b^2 + ab^3 + b^4\]

(ii) \(a^n - b^n\) is divisible by \(a + b\) if and only if \(n\) is even. For example,

\[(a^2 - b^2)/(a + b) = a - b\]

\[(a^6 - b^6)/(a - b) = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5\]

(iii) \(a^n + b^n\) is divisible by \(a + b\) if and only if \(n\) is odd. For example,

\[(a^3 + b^3)/(a + b) = a^2 - ab + b^2\]

\[(a^5 + b^5)/(a + b) = a^4 - a^3b + a^2b^2 - ab^3 + b^4\]

The three statements are easy consequences of the factor theorem, to be discussed later.
Exercises (As before, click on a problem inside a blue rectangle to see the answer:)

Write the following products:

1. \((6x - 7y)^2\)
2. \((x^n - y^n)^2\)
3. \((a + b + 1)^3\)
4. \((10 - x^3)^3\)
5. \((x^3 + y^3)(x^3 - y^3)\)
6. \((x^2 + x + 1)(x^2 - x - 1)\)
7. \((x^2 + x - 1)(x^2 - x + 1)\)

Find the following quotients:

8. \(\frac{9a^2 - b^2}{3a - b}\) Hint: This is of the form \((x^2 - y^2)/(x - y) = x + y\) with \(x = 3a, y = b\).
9. \(\frac{a^6 + b^3}{a^2 + b}\) Hint: This is of the form \((x^3 + y^3)/(x + y) = x^2 - xy + y^2\).
10. \(\frac{a^7 + x^7}{a + x}\)
11. \(\frac{a^4 - 100}{a^2 + 10}\) Hint: This is of the form \((x^2 - y^2)/(x + y) = x - y\).
12. \(\frac{x^8 - y^8}{x + y}\)

For what values of \(n\) are each of the following four quotients exact i.e. have 0 remainder? (Assume \(a \neq 0, b \neq 0,\) and \(n\) is an integer \(\geq 1,\) and that any denominator is not zero, as usual.)

13. \(\frac{a^n - b^n}{a - b}\)
14. \(\frac{a^n + b^n}{a + b}\)
15. \(\frac{a^{2n} + b^{2n}}{a + b}\)
16. \(\frac{a^{2n+1} + b^{2n+1}}{a + b}\)