A typical question on an exam asks you to find the derivative of a given function by using the definition. This means you need to compute
\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

Can you guess what the most common error is?

For each example, find the derivative of \( f(x) \) using the definition. If you can work all these problems without peeking at the solutions, you will most likely be well prepared for this topic on an exam.

1. \( f(x) = x^2 \)

   Answer:
   \[
   \frac{f(x + h) - f(x)}{h} = \frac{(x + h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h \to 2x \text{ as } h \to 0.
   \]

2. \( f(x) = x^3 + 2x \)

3. \( f(x) = \frac{1}{x} \)

   Answer:
   \[
   \frac{f(x + h) - f(x)}{h} = \frac{x + h}{x(x + h)} - \frac{x}{x} = \frac{x + h}{x(x + h)} - \frac{x}{x} = \frac{-h}{x(x + h)} = \frac{-1}{x(x + h)} \to \frac{-1}{x^2} \text{ as } h \to 0
   \]

4. \( f(x) = \frac{1}{2x + 1} \)
5. \( f(x) = \sqrt{x} \)

Answer:

\[
\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h} = \left( \frac{\sqrt{x + h} - \sqrt{x}}{h} \right) \left( \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} \right)
\]

\[
= \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} = \frac{-1}{\sqrt{x + h} + \sqrt{x}} \to -\frac{1}{2\sqrt{x}} \text{ as } h \to 0
\]

6. \( f(x) = \sqrt{2x + 1} \)

7. \( f(x) = \frac{1}{\sqrt{x}} \)

Answer to first question:

The most common error is to confuse the meanings of \( f(x + h) \) and \( f(x) + h \).
To find \( f(x + h) \) you replace the symbol \( x \) with the symbol \( x + h \) in the formula for \( f(x) \).
To find \( f(x) + h \) you first write down \( f(x) \) and then add \( h \) to that expression.
For example, if \( f(x) = x^2 + 2x \)

\[
f(x + h) = (x + h)^2 + 2(x + h) = x^2 + 2xh + h^2 + 2x + 2h
\]

and

\[
f(x) + h = x^2 + 2x + h
\]

Another common error is to cancel a term just because it occurs in the numerator and denominator. If a quantity is a (multiplicative) factor of the numerator and also the denominator, then the cancellation is valid. For example,

\[
\frac{f(a + b)}{f(c + d)} = \frac{a + b}{c + d}
\]

If a quantity is a summand then the cancellation is not valid in general. That is, the following is usually a blunder:

\[
\frac{f + a}{f + b} = \frac{a}{b}
\]

and it is very naughty to put such an egregious error on an exam.
Question: Suppose it is true that\[ \frac{x + a}{x + b} = \frac{a}{b} \]

What can you conclude?

Answer:
Suppose it is true that\[ \frac{x + a}{x + b} = \frac{a}{b} \]

Then

\[ a(x + b) = b(x + a) \]

\[ ax = bx \]

So

\[ x = 0 \text{ or } a = b \]

Answer to problem 2:

\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^3 + 2(x + h) - (x^3 + 2x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - (x^3 + 2x)}{h} = \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} = 3x^2 + 3xh + h^2 + 2 \rightarrow 3x^2 + 2 \text{ as } h \rightarrow 0
\]

Answer to problem 4:

\[
\frac{f(x + h) - f(x)}{h} = \sqrt{2(x + h) + 1} - \sqrt{2x + 1} = \frac{\sqrt{2(x + h) + 1} - \sqrt{2x + 1}}{h} \left( \frac{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}} \right) = \frac{2(x + h) + 1 - (2x + 1)}{h(\sqrt{2(x + h) + 1} + \sqrt{2x + 1})} = \frac{2h}{h(\sqrt{2(x + h) + 1} + \sqrt{2x + 1})} = \frac{2}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}
\]

\[ \rightarrow \frac{2}{2\sqrt{2x + 1}} = \frac{1}{\sqrt{2x + 1}} \text{ as } h \rightarrow 0 \]

The answers to problems 6 and 7 are on the next two pages:
\[
\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{2(x + h) + 1} - \sqrt{2x + 1}}{h} = \left( \frac{\sqrt{2(x + h) + 1} - \sqrt{2x + 1}}{h} \right) \left( \frac{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}} \right)
\]
\[
= \frac{2(x + h) + 1 - (2x + 1)}{h(\sqrt{2(x + h) + 1} + \sqrt{2x + 1})} = \frac{2h}{h(\sqrt{2(x + h) + 1} + \sqrt{2x + 1})} = \frac{2}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}
\]
\[
\rightarrow \frac{2}{2\sqrt{2x + 1}} = \frac{1}{\sqrt{2x + 1}} \text{ as } h \to 0
\]
\[
\frac{f(x + h) - f(x)}{h} = \frac{\frac{1}{\sqrt{x + h}} - \frac{1}{\sqrt{x}}}{h}
= \frac{\sqrt{x} - \sqrt{x + h}}{h(\sqrt{x + h}\sqrt{x})} \cdot \left(\frac{\sqrt{x} + \sqrt{x + h}}{\sqrt{x} + \sqrt{x + h}}\right)
= \frac{x - (x + h)}{h\sqrt{x + h}\sqrt{x}(\sqrt{x} + \sqrt{x + h})}
= \frac{-1}{\sqrt{x + h}\sqrt{x}(\sqrt{x} + \sqrt{x + h})}
\to \frac{-1}{2x\sqrt{x}} \text{ as } h \to 0.
\]