Linear Programming:

Problem:

A farmer has 150 acres of land on which he wants to grow cotton and potatoes. Government restrictions prevent the farmer from devoting more than 60 acres of land to cotton, but he can use as much acreage for potatoes as he wishes. It requires 30 inches per acre of water for irrigation to grow cotton and 15 inches per acre to grow potatoes. The farmer has 3000 acre-inches of water available. If the farmer’s profit per acre is $207 for cotton and $200 for potatoes, how many acres of cotton and how many acres of potatoes should he grow in order to obtain the largest possible profit from his land?

Comments: If there were no limitations on water or acres used for cotton, he would want to devote the entire 150 acres to cotton. You might think then that the strategy would be to devote 60 acres, the maximum allowed, to cotton and then use the other 90 acres for potatoes. But he must consider the water limitations: 60 acres of cotton would require $60 \times 30 = 1800$ acre-inches of water. On the other hand, 90 acres of potatoes needs $90 \times 15 = 1350$ acre-inches of water. Thus it would require $1800 + 1350 = 3150$ acre-inches of water. But the farmer has only 3000 acre-inches of water available for irrigation, so this scheme is impossible.

What if he plants 60 acres of cotton, so he uses $60 \times 30 = 1800$ acre-inches of the 3000 available, leaving 1200 acre-inches for potatoes. Since each acre of potatoes requires 15 inches of water, the farmer could plant $\frac{1200}{15} = 80$ acres of potatoes to use up the remaining water. Notice then that 10 acres would not be used in this scheme. With this scheme, his profit would be $60 \times 207 + 80 \times 200 = 28,400$.

The profit of $28,400 is not the highest profit he could make. For example, suppose the farmer planted 56 acres of cotton and 85 acres of potatoes. This requires a total of 141 acres, so there is enough land. There is also enough water available because 56 acres of cotton uses $56 \times 30 = 1680$ acre-inches and 85 acres of potatoes uses $85 \times 15 = 1275$ so $2955$ acre-inches would be used in all, which is possible. The total profit would be $56 \times 207 + 85 \times 200 = 28,592$, or $192$ more than by planting 60 acres of cotton and 80 acres of potatoes. Clearly, the farmer could even do better than that. There are 9 acres of land still available and 45 acre-inches of water unused. He could, for example, plant 3 more acres of potatoes to use up the available water and thus increase his profits by an additional $600.
The problem stated mathematically (as a linear programming problem):

Let \( x \) denote the number of acres of cotton and \( y \) the number of acres of potatoes planted.

Maximize the profit \( P = 207x + 200y \)

Subject to the following constraints:

\[
\begin{align*}
x + y & \leq 150 \\
x & \leq 60 \\
30x + 15y & \leq 3000 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

The last two inequalities are just mathematically stating the obvious: the number of acres planted is non-negative. First we sketch the region of possible solutions, which was done last time:
Next, we forget temporarily the constraints and sketch level curves $P = 207x + 200y$ for a few values of $P$:
For any solution to the maximization problem with constraints, the values of x and y must give a point (x, y) that lies inside the region of possible solutions. We don’t know the largest possible value of P just yet, but we notice if we guess at a value of P that is too large, then there is no value of x and y for that P that lies within the region of possible regions. If we guess at a P that is too small, we notice that P can be increased. Finally, notice that the largest (or smallest) value of P with a solution (x, y) inside the region will occur at a corner.

So in order to find the largest possible P, we need only check the value of P at each of the corners:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>P = 207x + 200y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>150</td>
<td>30,000</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
<td>12,420</td>
</tr>
<tr>
<td>60</td>
<td>80</td>
<td>28,420</td>
</tr>
<tr>
<td>50</td>
<td>100</td>
<td>30,350</td>
</tr>
</tbody>
</table>

We see that the profit is maximized by choosing 50 acres for cotton and 100 acres for potatoes.

Summary: Given a linear programming problem, first sketch the region of possible solutions. Then check the values of the objective function (the profit, the cost, etc.) at all the corners. The largest value will give the largest possible value of the objective function subject to the constraints. Similarly, the smallest value will give the least possible value of the objective function subject to the constraints.