Midterm Exam 1
Math 132-06, Fall 2005

You have 50 minutes. No notes, no books, no calculators. You must show all work to receive credit! Good luck!

Name: Solutions

ID #: ________________________________

1. _________ (/40 points)

2. _________ (/15 points)

3. _________ (/25 points)

4. _________ (/20 points)

Total _________ (/100 points)
1. [40 points] Evaluate the following limits if they exist. If the limit does not exist, explain why. Justify your answers using the limit laws or facts about continuity.

(a) \[ \lim_{x \to 3} \frac{x - 3}{x^2 - 2x - 3} = \lim_{x \to 3} \frac{x - 3}{(x - 3)(x + 1)} \]
\[ = \lim_{x \to 3} \frac{1}{x + 1} \]
\[ = \frac{1}{4} \]

(b) \[ \lim_{x \to -1} \frac{x - 3}{x^2 - 2x - 3} = \lim_{x \to -1} \frac{1}{x + 1}, \text{ as above.} \]

This limit does not exist, because the function approaches \( +\infty \) from the right and \(-\infty \) from the left.

(c) \[ \lim_{x \to 0^+} \frac{\sin(2\sqrt{x})}{\sqrt{x}} = \lim_{\theta \to 0^+} \frac{\sin(2\theta)}{\theta} = \lim_{\theta \to 0^+} \frac{2 \sin(2\theta)}{2 \theta} \]
\[ = 2 \cdot 1 = 2 \]

(d) \[ \lim_{x \to \infty} \frac{\sqrt{x} + 5}{3\sqrt{x} - 2} = \lim_{x \to \infty} \frac{\sqrt{x} + 5}{3\sqrt{x} - 2} \]
\[ = \lim_{x \to \infty} \frac{1 + 5\sqrt{1/x}}{3 - 2\sqrt{1/x}} \]
\[ = \frac{1}{3} \]
2. [15 points] Evaluate \( \lim_{x \to 0^+} \sqrt{x} \cos(1/x) \), using the Sandwich Theorem.

\[-1 \leq \cos \left( \frac{1}{x} \right) \leq 1 \quad \text{for all } x.
\]

So \(-\sqrt{x} \leq \sqrt{x} \cos \left( \frac{1}{x} \right) \leq \sqrt{x} \) for all \( x > 0 \).

Thus, by the Sandwich Theorem,

\[
\lim_{x \to 0^+} -\sqrt{x} \leq \lim_{x \to 0^+} \sqrt{x} \cos \left( \frac{1}{x} \right) \leq \lim_{x \to 0^+} \sqrt{x}
\]

\[0 \leq \lim_{x \to 0^+} \sqrt{x} \cos \left( \frac{1}{x} \right) \leq 0.\]

So \( \lim_{x \to 0^+} \sqrt{x} \cos \left( \frac{1}{x} \right) = 0 \).
3. [25 points] This problem uses the following graph of $f(x)$.

(a) [9 points] Compute $\lim_{x \to 0^-} f(x)$, $\lim_{x \to -1^+} f(x)$, and $\lim_{x \to 1} f(x)$.

\[
\lim_{x \to 0^-} f(x) = \infty.
\]
\[
\lim_{x \to -1^+} f(x) = 1.
\]
\[
\lim_{x \to 1} f(x) = 2.
\]

(b) [6 points] At what $x$-values is $f$ discontinuous?

At $x = -1$, 0, and 1.

(c) [10 points] What are the horizontal and vertical asymptotes?

Horizontal asymptotes at $y = 0$, $y = 2$.

Vertical asymptote at $x = 0$. 
4. [20 points]

(a) [8 points] State the Intermediate Value Theorem.
Let \( f(x) \) be a continuous function on \([a, b]\), and let \( N \) be any number between \( f(a) \) and \( f(b) \). Then there is an \( x \)-value \( c \), in the interval \([a, b]\), such that \( f(c) = N \).

Decide whether each of the following statements is true or false, and explain your reasoning. You must write out the word “True” or “False” for each one.

(b) [6 points] The function \( f(x) = x - \cos(x) \) has a root between \( x = 0 \) and \( x = \frac{\pi}{2} \).

\[ f(0) = 0 - \cos(0) = -1. \]
\[ f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}. \]

True. \( f(x) \) is continuous, because it is the difference of two continuous functions. Thus, by the Intermediate Value Theorem, \( f(c) = 0 \) for some \( c \) in \([0, \frac{\pi}{2}]\).

(c) [6 points] The function \( g(x) = (x + 2) \frac{|x|}{x} \) has a root between \( x = -1 \) and \( x = 1 \).

\[ g(-1) = (-1 + 2) \cdot \frac{|-1|}{-1} = 1 \cdot -1 = -1. \]
\[ g(1) = (1 + 2) \cdot \frac{|1|}{1} = 3 \cdot 1 = 3. \]

False. \( g(x) \) is not continuous, so IVT doesn’t apply. In fact, the only root of \( g(x) \) occurs when \( x + 2 = 0 \), i.e. at \( x = -2 \).