Turaev-Viro invariants, colored Jones polynomials and volume

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Abstract

We obtain a formula for the Turaev-Viro invariants of a link complement in terms of values of the colored Jones polynomial of the link. As an application we give the first examples of 3-manifolds where the "large r" asymptotics the Turaev-Viro invariants determine the hyperbolic volume. We verify the volume conjecture of Chen and the third named author [6] for the Figure-eight knot and the Borromean rings. Our calculations also exhibit new phenomena of asymptotic behavior of values of the colored Jones polynomials that seem not to be predicted by neither the Kashaev-Murakami-Murakami volume conjecture and various of its generalizations nor by Zagier's quantum modularity conjecture. We then conjecture that the asymptotics of the Turaev-Viro invariants of any link complement determine the simplicial volume of the link, and verify it for all knots with zero simplicial volume. Finally we observe that our simplicial volume conjecture is stable under connect sum and split unions of links.

1 Introduction

In [33], Turaev and Viro defined a family of 3-manifold invariants as state sums on triangulations of manifolds. The family is indexed by an integer r, and for each r the invariant depends on a choice of a $2r$-th root of unity. In the last couple of decades these invariants have been refined and generalized in many directions and shown to be closely related to the Witten-Reshetikhin-Turaev invariants. (See [1, 15, 32, 20] and references therein.) Despite these efforts, the relationship between the Turaev-Viro invariants and the geometric structures on 3-manifolds arising from Thurston's geometrization picture is not understood. Recently Chen and the third named author [6] conjectured that, evaluated at appropriate roots of unity, the asymptotic behavior of the Turaev-Viro invariants of a complete hyperbolic 3-manifold, of finite volume, determines the hyperbolic volume of the manifold, and presented compelling experimental evidence to their conjecture.

In the present paper we focus mostly on the Turaev-Viro invariants of link complements in S^3. Our main result gives a formula of the Turaev-Viro invariants of a link complement in terms of values of the colored Jones polynomial of the link. Using the formula we rigorously verify the volume conjecture of [6] for the Figure-eight knot and Borromean rings complement. These are to the best of the authors knowledge the first examples of this kind. Our calculations exhibit new phenomena of asymptotic behavior of the colored Jones polynomial that does not seem to be predicted by the volume conjectures [14, 25, 7] or by Zagier's quantum modularity conjecture [37].

1.1 Relationship between knot invariants

To state our results we need to introduce some notations. For a link $L \subset S^3$, let $TV_r(S^3 \smallsetminus L, q)$ denote the r-th Turaev-Viro invariant of the link complement evaluated at the root of unity q such that q^2 is

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primitive of degree r. Throughout this paper, we will consider the case that $q = A^2$, where A is either a primitive $4r$-th root for any integer r or a primitive $2r$-th root for any odd integer r.

We use the notation $\mathbf{i} = (i_1, \dots, i_n)$ for a multi-integer of n components (an n-tuple of integers) and use the notation $1 \leqslant \mathbf{i} \leqslant m$ to describe all such multi-integers with $1 \leqslant i_k \leqslant m$ for each $k \in \{1, \dots, n\}$. Given a link L with n components, let $J_{L,\mathbf{i}}(t)$ denote the $\mathbf{i}$-th colored Jones polynomial of L whose k-th component is colored by i_k [19, 17]. If all the components of L are colored by the same integer i, then we simply denote $J_{L,(i,\dots,i)}(t)$ by $J_{L,i}(t)$. If L is a knot, then $J_{L,i}(t)$ is the usual i-th colored Jones polynomial. The polynomials are indexed so that $J_{L,1}(t) = 1$ and $J_{L,2}(t)$ is the ordinary ones polynomial, and are normalized so that

$$J_{U,i}(t) = [i] = \frac{A^{2i} - A^{-2i}}{A^2 - A^{-2}}$$

for the unknot U, where by convention $t = A^4$. Finally we define

$$\eta_r = \frac{A^2 - A^{-2}}{\sqrt{-2r}} \text{ and } \eta'_r = \frac{A^2 - A^{-2}}{\sqrt{-r}}.$$

Before stating our main result, let us recall once again the convention that $q = A^2$ and $t = A^4$.

Theorem 1.1. *Let L be a link in S^3 with n components.*

(1) For an integer $r \geqslant 3$ and a primitive $4r$-th root of unity A, we have

$$TV_r(S^3 \smallsetminus L, q) = \eta_r^2 \sum_{1 \leqslant \mathbf{i} \leqslant r-1} |J_{L,\mathbf{i}}(t)|^2.$$

(2) For an odd integer $r \geqslant 3$ and a primitive $2r$-th root of unity A, we have

$$TV_r(S^3 \smallsetminus L, q) = 2^{n-1} (\eta'_r)^2 \sum_{1 \leqslant \mathbf{i} \leqslant \frac{r-1}{2}} |J_{L,\mathbf{i}}(t)|^2.$$

Extending an earlier result of Roberts [29], Beneditti and Petronio [1] showed that the invariants $TV_r(M, e^{\frac{\pi i}{r}})$ of a 3-manifold M with boundary coincide up to a scalar with the Witten-Reshetikhin-Turaev invariants of the double of M. In the proof of Theorem 1.1, we first apply Beneditti-Petronio's argument to the case that r is odd and A is a primitive $2r$-th root, extending this relation to the Turaev-Viro invariants and the $SO(3)$ Reshetikhin-Turaev invariants [17, 3, 4]. See Theorem 3.1. Then the rest of the proof follows from the properties of the Reshetikhin-Turaev topological quantum field theory as established by Blanchet, Habegger, Masbaum and Vogel [2, 4].

Note that for any primitive r-th root of unity with $r \geqslant 3$, the quantities η_r and η'_r are real and non-zero. Since $J_{L,1}(t) = 1$, and with the notation as in Theorem 1.1, we have $TV_r(S^3 \smallsetminus L, q) \geqslant \eta_r^2$ in case (1) and $TV_r(S^3 \smallsetminus L, q) \geqslant 2^{n-1}(\eta'_r)^2$ in case (2). In particular, we have the following

Corollary 1.2. *For any $r \geqslant 3$, any root $q = A^2$ and any link L in S^3, we have*

$$TV_r(S^3 \smallsetminus L, q) > 0.$$

We note that the values of the colored Jones polynomials do not have such a positivity property. In fact, all the values that are involved in the Kashaev volume conjecture [14, 25] are known to vanish for the split links and the Whitehead chains [25, 35].

Another immediate consequence of Theorem 1.1 is that links with the same colored Jones polynomials have the same Turaev-Viro invariants. In particular, since the colored Jones polynomial is invariant under Conway mutations and the genus 2 mutations [23], we obtain the following.

Corollary 1.3. *For any $r \geqslant 3$, any root $q = A^2$ and any link L in S^3, the invariants $TV_r(S^3 \smallsetminus L, q)$ remain unchanged under Conway mutations and the genus 2 mutations.*

1.2 Asymptotics of Turaev-Viro and colored Jones link invariants

We are particularly interested in the large r asymptotics of the invariants $TV_r(S^3 \smallsetminus L, A^2)$ in the case that either $A = e^{\frac{\pi i}{2r}}$ for integers $r \geqslant 3$ or $A = e^{\frac{\pi i}{r}}$ for odd integers $r \geqslant 3$. With these choices of A, we have in the former case that

$$\eta_r = \frac{2\sin(\frac{\pi}{r})}{\sqrt{2r}},$$

and in the latter case that

$$\eta'_r = \frac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}.$$

In [6], Chen and the third named author presented compelling experimental evidence to the following

Conjecture 1.4. *[6] For every hyperbolic 3-manifold M, we have*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M, e^{\frac{2\pi i}{r}})) = Vol(M),$$

where r runs over all odd integers.

Conjecture 1.4 impies that $TV_r(M, e^{\frac{2\pi i}{r}})$ grows exponentially in r. This is particularly surprising since the corresponding growth of $TV_r(M, e^{\frac{\pi i}{r}})$ is expected, and in many cases known, to be only polynomially by Witten's asymptotic expansion conjecture [36, 13]. For closed 3-manifolds, this polynomial growth was established by Garoufalidis [8]. Combining [8, Theorem 2.2] and the results of [1], one has that for every 3-manifold M with nonempty boundary, there exist constants $C > 0$ and N such that $|TV_r(M, e^{\frac{\pi i}{r}})| \leqslant Cr^N$. This together with Theorem 1.1 imply the following.

Corollary 1.5. *For any link L in S^3, there exist constants $C > 0$ and N such that for any integer r and multi-integer $\mathbf{i}$ with $1 \leqslant \mathbf{i} \leqslant r-1$, the value of the $\mathbf{i}$-th colored Jones polynomial at $t = e^{\frac{2\pi i}{r}}$ satisfies*

$$|J_{L,\mathbf{i}}(e^{\frac{2\pi i}{r}})| \leqslant Cr^N.$$

Hence, $J_{L,\mathbf{i}}(e^{\frac{2\pi i}{r}})$ grows at most polynomially in r.

As a main application of Theorem 1.1, we provide the first rigorous evidences to Conjecture 1.4.

Theorem 1.6. *Let L be either the Figure-eight knot or the Borromean rings, and let M be the complement of L in S^3. Then*

$$\lim_{r\to+\infty} \frac{2\pi}{r} \log TV_r(M, e^{\frac{2\pi i}{r}}) = \lim_{m\to+\infty} \frac{4\pi}{2m+1} \log|J_{L,m}(e^{\frac{4\pi i}{2m+1}})| = Vol(M),$$

where $r = 2m+1$ runs over all odd integers.

The asymptotic behavior of the values of $J_{L,m}(t)$ at $t = e^{\frac{2\pi i}{m+\frac{1}{2}}}$ was not predicted previously by either the original volume conjecture [14, 25] or any of its generalizations [7, 24]. Theorem 1.6 seems to suggest that these values grow exponentially in m with growth rate the hyperbolic volume. This is somewhat surprising because as noted in [9], and also in Corollary 1.5, that for any positive integer l, $J_{L,m}(e^{\frac{2\pi i}{m+l}})$ grows only polynomially in m.

Question 1.7. *Is it true that for any hyperbolic link L,*

$$\lim_{m\to+\infty} \frac{2\pi}{m} \log|J_{L,m}(e^{\frac{2\pi i}{m+\frac{1}{2}}})| = Vol(S^3 \smallsetminus L)?$$

1.3 Knots with zero simplicial volume

Recall that the simplicial volume $||L||$ of a link L is the sum of the volumes of the hyperbolic pieces of the geometric decomposition of the link complement, divided by the volume of the regular ideal hyperbolic tetrahedron. In particular, if the geometric decomposition has no hyperbolic pieces, then $||L|| = 0$. As a natural generalization of Conjecture 1.4, one can conjecture that for every link L the asymptotics of $TV_r(S^3\smallsetminus L, e^{\frac{2\pi i}{r}})$ determines $||L||$. See Conjecture 5.1.

Using Theorem 1.1 and the positivity of the Turaev-Viro invariants (Corollary 1.2), we have a proof of Conjecture 5.1 for the knots with zero simplicial volume.

Theorem 1.8. *Let $K \subset S^3$ be a knot with simplicial volume zero. Then*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log TV_r(S^3\smallsetminus K, e^{\frac{2\pi i}{r}}) = ||K|| = 0,$$

where r runs over all odd integers.

We also observe that, unlike the original volume conjecture that is not true for split links [25, Remark 5.3], Conjecture 5.1 is closed under the split unions of links, and under some assumptions is also closed under the connected sums.

1.4 Organization

The paper is organized as follows. In Subsection 2.1, we review the Reshetikhin-Turaev invariants [28] following the skein theoretical approach by Blanchet, Habegger, Masbaum and Vogel [2, 3, 4]. In Subsection 2.2, we recall the definition of the Turaev-Viro invariants, and consider an $SO(3)$-version of them for the purpose of extending Beneditti-Petronio's theorem [1] to other roots of unity (Theorem 3.1). The relationship between the two versions of the Turaev-Viro invariants is given in Theorem 2.9 whose proof is postponed to the Appendix. We prove Theorem 1.1 in Section 3, and prove Theorem 1.6 and Theorem 1.8 respectively in Sections 4 and 5.

1.5 Acknowledgement

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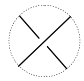
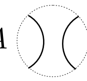
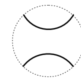
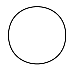
2 Preliminaries

2.1 Reshetikhin-Turaev invariants and TQFTs

In this subsection we review the definition and basic properties of the Reshetikhin-Turaev invariants. Our exposition follows the skein theoretical approach of Blanchet, Habegger, Masbaum and Vogel [2, 3, 4].

A framed link in an oriented 3-manifold M is a smooth embedding L of a disjoint union of finitely many thickened circles $S^1 \times [0, \epsilon]$, for some $\epsilon > 0$, into M. Let $\mathbb{Z}[A, A^{-1}]$ be the ring of Laurent polynomials in the indeterminate A. Then following [27, 31], the *Kauffman bracket skein module* $K_A(M)$ of M is defined as the quotient of the free $\mathbb{Z}[A, A^{-1}]$-module generated by the isotopy classes of framed links in M by the following two relations:

(1) *Kauffman Bracket Skein Relation:*  $= A$  $+ \; A^{-1}$ .

(2) *Framing Relation:* $L \cup$  $= (-A^2 - A^{-2})\, L.$

There is a canonical isomorphism

$$\langle \; \rangle : K_A(S^3) \to \mathbb{Z}[A, A^{-1}]$$

between the Kauffman bracket skein module of S^3 and $\mathbb{Z}[A, A^{-1}]$ viewed a module over itself. The Laurent polynomial $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ determined by the framed link $L \subset S^3$ is called the *Kauffman bracket* of L.

The Kauffman bracket skein module $K_A(T)$ of the solid torus $T = D^2 \times S^1$ is canonically isomorphic to the module $\mathbb{Z}[A, A^{-1}][z]$. Here we consider D^2 as the unit disk in the complex plane, and call the framed link $[0, \epsilon] \times S^1 \subset D^2 \times S^1$, for some $\epsilon > 0$, the core of T. Then the isomorphism above is given by sending i parallel copies of the core of T to z^i. A framed link L in S^3 of n components defines an $\mathbb{Z}[A, A^{-1}]$-multilinear map

$$\langle \quad , \ldots , \quad \rangle_L : K_A(T)^{\otimes n} \to \mathbb{Z}[A, A^{-1}]$$

on $K_A(T)$, called the *Kauffman multi-bracket*, as follows. For monomials $z^{i_k} \in \mathbb{Z}[A, A^{-1}][z] \cong K_A(T)$, $k = 1, \ldots, n$, let $L(z^{i_1}, \ldots, z^{i_n})$ be the framed link in S^3 obtained by cabling the k-th component of L by i_k parallel copies of the core. Then define

$$\langle z^{i_1}, \ldots, z^{i_n} \rangle_L \doteq \langle L(z^{i_1}, \ldots, z^{i_n}) \rangle,$$

and extend $\mathbb{Z}[A, A^{-1}]$-multilinearly to the whole $K_A(T)$. For the unknot U and any polynomial $P(z) \in \mathbb{Z}[A, A^{-1}][z]$, we simply denote the bracket $\langle P(z) \rangle_U$ by $\langle P(z) \rangle$.

The i-th Chebyshev polynomial $e_i \in \mathbb{Z}[A, A^{-1}][z]$ is defined by the recurrence relations $e_0 = 1$, $e_1 = z$, and $z e_j = e_{j+1} + e_{j-1}$, and satisfies

$$\langle e_i \rangle = (-1)^i [i + 1].$$

The colored Jones polynomials of an oriented knot K in S^3 are defined using e_i as follows. Let D be a diagram of K with writhe number $w(D)$, and frame D with the blackboard framing. Then the $(i+1)$-st colored Jones polynomial of K is

$$J_{K,i+1}(t) = \left((-1)^i A^{i^2 + 2i} \right)^{w(D)} \langle e_i \rangle_D.$$

The colored Jones polynomials for an oriented link L in S^3 is defined similarly. Let D be a diagram of L with writhe number $w(D)$ and the blackboard framing. For a multi-integer $\mathbf{i} = (i_1, \ldots, i_n)$, let $\mathbf{i} + \mathbf{1} = (i_1 + 1, \ldots, i_n + 1)$. Then the $(\mathbf{i} + \mathbf{1})$-st colored Jones polynomial of L is defined by

$$J_{L,\mathbf{i}+\mathbf{1}}(t) = \left((-1)^{\sum_{k=1}^{n} i_k} A^{s(\mathbf{i})} \right)^{w(D)} \langle e_{i_1}, \ldots, e_{i_n} \rangle_D,$$

where $s(\mathbf{i}) = \sum_{k=1}^{n} (i_k^2 + i_k)$.

We note that a change orientation on some or all components of L changes the writhe number of D, and changes $J_{L,\mathbf{i}}(t)$ only by a power of A. Therefore, for an unoriented link L and a complex number A with $|A| = 1$, the modulus of the value of $J_{L,\mathbf{i}}(t)$ at $t = A^4$ is well defined, and

$$|J_{L,\mathbf{i}}(t)| = |\langle e_{i_1 - 1}, \ldots, e_{i_n - 1} \rangle_D|. \tag{2.1}$$

If M is a closed oriented 3-manifold obtained by doing surgery along a framed link L in S^3, then the specialization of the Kauffman multi-bracket at roots of unity yields invariants of 3-manifolds. From now on, let A be either a primitive $4r$-th root of unity for an integer $r \geqslant 3$ or a primitive $2r$-th root of unity for an odd integer $r \geqslant 3$. To define the Reshetikhin-Turaev invariants, we need to recall some special elements of $K_A(T) \cong \mathbb{Z}[A, A^{-1}][z]$, called the *Kirby coloring*, defined by

$$\omega_r = \sum_{i=0}^{r-2} \langle e_i \rangle e_i$$

for any integer r, and

$$\omega'_r = \sum_{i=0}^{m-1} \langle e_{2i} \rangle e_{2i}$$

for any odd integer $r = 2m+1$. We also for any r introduce

$$\kappa_r = \eta_r \langle \omega_r \rangle_{U_+},$$

and for any odd r introduce

$$\kappa'_r = \eta'_r \langle \omega'_r \rangle_{U_+},$$

where U_+ is the unknot with framing 1.

Definition 2.1. *Let M be a closed oriented 3-manifold obtained from S^3 by doing surgery along a framed link L with number of components $n(L)$ and signature $\sigma(L)$.*

(1) Then the Reshetikhin-Turaev invariant of the M is defined by

$$\langle M \rangle_r = \eta_r^{1+n(L)} \, \kappa_r^{-\sigma(L)} \, \langle \omega_r, \ldots, \omega_r \rangle_L$$

for any integer $r \geqslant 3$, and by

$$\langle M \rangle'_r = (\eta'_r)^{1+n(L)} \, (\kappa'_r)^{-\sigma(L)} \, \langle \omega'_r, \ldots, \omega'_r \rangle_L$$

for any odd integer $r \geqslant 3$.

(2) Let also L' be a framed link in M. Then the Reshetikhin-Turaev invariant of the pair (M, L') is defined by

$$\langle M, L' \rangle_r = \eta_r^{1+n(L)} \, \kappa_r^{-\sigma(L)} \, \langle \omega_r, \ldots, \omega_r, 1 \rangle_{L \cup L'}$$

for any integer $r \geqslant 3$, and by

$$\langle M, L' \rangle'_r = (\eta'_r)^{1+n(L)} \, (\kappa'_r)^{-\sigma(L)} \, \langle \omega'_r, \ldots, \omega'_r, 1 \rangle_{L \cup L'}$$

for any odd integer $r \geqslant 3$.

Remark 2.2. (1) We will call $\langle M \rangle'_r$ the $SO(3)$ Reshetikhin-Tureav invariant of M.

(2) For any element S in $K_A(M)$ represented by a $\mathbb{Z}[A, A^{-1}]$-linear combinations of framed links in M, one can define $\langle M, S \rangle_r$ and $\langle M, S \rangle'_r$ by $\mathbb{Z}[A, A^{-1}]$-linear extensions.

(3) Since S^3 is obtained by doing surgery along the empty link, we have $\langle S^3 \rangle_r = \eta_r$ and $\langle S^3 \rangle'_r = \eta'_r$. Moreover, for any link $L \subset S^3$ we have

$$\langle S^3, L \rangle_r = \eta_r \, \langle L \rangle, \ \text{ and } \ \langle S^3, L \rangle'_r = \eta'_r \, \langle L \rangle.$$

In [4], Blanchet, Habegger, Masbaum and Vogel also constructed the underling topological quantum field theories Z_r and $Z_{r'}$ of the Reshetikhin-Turaev invariants, which can be summarized as follows.

Theorem 2.3. *[4, Theorem 1.4]*

(1) Let Σ be a closed oriented surface, then for any integer $r \geqslant 3$, there exists a finite dimensional $\mathbb{C}$-vector space $Z_r(\Sigma)$ satisfying

$$Z_r(\Sigma_1 \coprod \Sigma_2) \cong Z_r(\Sigma_1) \otimes Z_r(\Sigma_2),$$

and for each odd integer $r \geqslant 3$, there exists a finite dimensional $\mathbb{C}$-vector space $Z'_r(\Sigma)$ satisfying

$$Z'_r(\Sigma_1 \coprod \Sigma_2) \cong Z'_r(\Sigma_1) \otimes Z'_r(\Sigma_2).$$

(2) If H is a handlebody with $\partial H = \Sigma$, then $Z_r(\Sigma)$ and $Z'_r(\Sigma)$ are respectively quotients of the Kauffman bracket skein module $K_A(H)$.

(3) Every compact oriented 3-manifold M with $\partial M = \Sigma$ and a framed link L in M defines for any integer r a vector $Z_r(M,L)$ in $Z_r(\Sigma)$, and for any odd integer r a vector $Z'_r(M,L)$ in $Z'_r(\Sigma)$.

(4) For for any integer r, there is a sesquilinear pairing $\langle\ ,\ \rangle$ on $Z_r(\Sigma)$ with the following property: Given oriented 3-manifolds M_1 and M_2 with boundary $\Sigma = \partial M_1 = \partial M_2$, and framed links $L_1 \subset M_1$ and $L_2 \subset M_2$, we have

$$\langle M, L\rangle_r = \langle Z_r(M_1, L_1), Z_r(M_2, L_2)\rangle,$$

where $M = M_1 \bigcup_\Sigma (-M_2)$ is the closed 3-manifold obtained by gluing M_1 and M_2 along Σ and $L = L_1 \coprod L_2$. Similarly, for any odd integer r, there is a sesquilinear pairing $\langle\ ,\ \rangle$ on $Z'_r(\Sigma)$, such tor any M and L as above,

$$\langle M, L\rangle'_r = \langle Z'_r(M_1, L_1), Z'_r(M_2, L_2)\rangle.$$

For the purpose of this paper, we will only need to understand the TQFT vector spaces of the torus $Z_r(T^2)$ and $Z'_r(T^2)$. These vector spaces are quotients of $K_A(T) \cong \mathbb{Z}[A, A^{-1}][z]$, hence the Chebyshev polynomials $\{e_i\}$ define vectors in $Z_r(T^2)$ and $Z'_r(T^2)$. We have the following

Theorem 2.4. *[4, Corollary 4.10, Remark 4.12]*

(1) For any integer $r \geqslant 3$, the vectors $\{e_0, \dots, e_{r-2}\}$ form a Hermitian basis of $Z_r(T^2)$.

(2) For any odd integer $r = 2m+1$, the vectors $\{e_0, \dots, e_{m-1}\}$ form a Hermitian basis of $Z'_r(T^2)$.

(3) In $Z'_r(T^2)$, we have for any i with $0 \leqslant i \leqslant m-1$ that

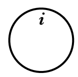
$$e_{m+i} = e_{m-1-i}. \tag{2.2}$$

Therefore, the vectors $\{e_{2i}\}_{i=0,\dots,m-1}$ also form a Hermitian basis of $Z'_r(T^2)$.

2.2 Turaev-Viro invariants

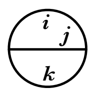
In this subsection, we recall the definition and basic properties of the Turaev-Viro invariants [33, 15]. The approach of [33] relies on quantum $6j$-symbols while the definition of Kauffman and Lins [15] uses invariants of spin networks. The two definitions were shown to be equivalent in [26]. The formalism of [15] turns out to be more convenient to work with when using skein theoretic techniques to relate the Turaev-Viro invariants to the Reshetikhin-Turaev invariants.

For an integer $r \geqslant 3$, let $I_r = \{0, 1, \ldots, r-2\}$ be the set of non-negative integers less than or equal to $r-2$. Let q be a $2r$-th root of unity such that q^2 is a primitive r-th root. For example, $q = A^2$, where A is either a primitive $4r$-th root or for odd r a primitive $2r$-th root, satisfies the condition. For $i \in I_r$, define



$$= (-1)^i [i+1].$$

A triple (i, j, k) of elements of I_r is called *admissible* if (1) $i + j \geqslant k$, $j + k \geqslant i$ and $k + i \geqslant j$, (2) $i + j + k$ is an even, and (3) $i + j + k \leqslant 2(r-2)$. For an admissible triple (i, j, k), define



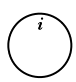
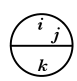
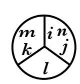
$$= (-1)^{-\frac{i+j+k}{2}} \frac{[\frac{i+j-k}{2}]![\frac{j+k-i}{2}]![\frac{k+i-j}{2}]![\frac{i+j+k}{2}+1]!}{[i]![j]![k]!}.$$

A 6-tuple (i, j, k, l, m, n) of elements of I_r is called *admissible* if the triples (i, j, k), (j, l, n), (i, m, n) and (k, l, m) are admissible. For an admissible 6-tuple (i, j, k, l, m, n), define

$$= \frac{\prod_{a=1}^{4}\prod_{b=1}^{3}[Q_b - T_a]!}{[i]![j]![k]![l]![m]![n]!} \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z [z+1]!}{\prod_{a=1}^{4}[z - T_a]! \prod_{b=1}^{3}[Q_b - z]!},$$

where

$$T_1 = \frac{i+j+k}{2}, \quad T_2 = \frac{i+m+n}{2}, \quad T_3 = \frac{j+l+n}{2}, \quad T_4 = \frac{k+l+m}{2},$$
$$Q_1 = \frac{i+j+l+m}{2}, \quad Q_2 = \frac{i+k+l+n}{2}, \quad Q_3 = \frac{j+k+m+n}{2}.$$

Remark 2.5. The symbols ,  and , used above, are examples of spin networks: trivalent ribbon graphs with ends colored by integers. The expressions on the right hand sides of above equations give the Kauffman bracket invariant of the corresponding networks. See [15, Chapter 9]. In the language of [15], the second and third spin networks above are the trihedral and tetrahedral networks, denoted by $\theta(i, j, k)$ and $\tau(i, j, k)$ therein, and the corresponding invariants are the trihedral and tetrahedral coefficients, respectively.

Definition 2.6. *A* coloring *of a Euclidean tetrahedron Δ is an assignment of elements of I_r to the edges of Δ, and is* admissible *if the triple of elements of I_r assigned to the three edges of each face of Δ is admissible. See Figure 1 for a geometric interpretation of tetrahedral coefficients.*

Let $\mathcal{T}$ be a triangulation of M. If M is with non-empty boundary, then we let $\mathcal{T}$ be an ideal triangulation of M, i.e., a gluing of finitely many truncated Euclidean tetrahedra by affine homeomorphisms between pairs of faces. In this way, there are no vertices, and instead, the triangles coming from truncations form a triangulation of the boundary of M. By edges of an ideal triangulation, we only mean the ones coming from the edges of the tetrahedra, not the ones from the truncations. A *coloring at level* r of

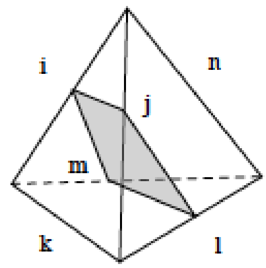
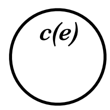
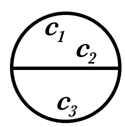


Figure 1: The quantities $T_1, \dots, T_4$ correspond to faces and Q_1, Q_2, Q_3 correspond to quadrilaterals.

the triangulated 3-manifold $(M, \mathcal{T})$ is an assignment of elements of I_r to the edges of $\mathcal{T}$, and is *admissible* if the 6-tuple assigned to the edges of each tetrahedron of $\mathcal{T}$ is admissible. Let c be an admissible coloring of $(M, \mathcal{T})$ at level r. For each edge e of $\mathcal{T}$, let

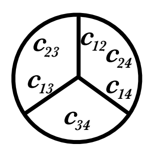
$|e|_c =$  .

For each face f of $\mathcal{T}$ with edges e_1, e_2 and e_3, let

$|f|_c =$  ,

where $c_i = c(e_i)$.

For each tetrahedra Δ in $\mathcal{T}$ with vertices $v_1, \dots, v_4$, denote by e_{ij} the edge of Δ connecting the vertices v_i and v_j, $\{i, j\} \subset \{1, \dots, 4\}$, and let

$|\Delta|_c =$  ,

where $c_{ij} = c(e_{ij})$.

Definition 2.7. *Let A_r be the set of admissible colorings of $(M, \mathcal{T})$ at level r, and let V, E F and T respectively be the sets of vertices, edges, faces and tetrahedra in $\mathcal{T}$. Then the r-th Turaev-Viro invariant is defined by*

$$TV_r(M) = \eta_r^{2|V|} \sum_{c \in A_r} \frac{\prod_{e \in E} |e|_c \prod_{\Delta \in T} |\Delta|_c}{\prod_{f \in E} |f|_c}.$$

For an odd integer $r \geqslant 3$, one can also consider an $SO(3)$-version of the Turaev-Viro invariants $TV'_r(M)$ of M, which will relate $TV_r(M)$ and the Reshetkin-Turaev invariants $\langle D(M) \rangle_r$ of the double of M (Theorems 2.9, 3.1). The invariant $TV'_r(M)$ is defined as follows. Let $I'_r = \{0, 2, \dots, r-5, r-3\}$ be the set of non-negative even integers less than or equal to $r-2$. An $SO(3)$-*coloring* of a Euclidean tetrahedron Δ is an assignment of elements of I'_r to the edges of Δ, and is *admissible* if the triple assigned to the three edges of each face of Δ is admissible. Let $\mathcal{T}$ be a triangulation of M. An $SO(3)$-*coloring at level* r of the triangulated 3-manifold $(M, \mathcal{T})$ is an assignment of elements of I'_r to the edges of $\mathcal{T}$, and is *admissible* if the 6-tuple assigned to the edges of each tetrahedron of $\mathcal{T}$ is admissible.

Definition 2.8. *Let A'_r be the set of $SO(3)$-admissible colorings of $(M, \mathcal{T})$ at level r. Define*

$$TV'_r(M) = (\eta'_r)^{2|V|} \sum_{c \in A'_r} \frac{\prod_{e \in E} |e|_c \prod_{\Delta \in T} |\Delta|_c}{\prod_{f \in E} |f|_c}.$$

The relationship between $TV_r(M)$ and $TV'_r(M)$ is given by the following theorem.

Theorem 2.9. *Let M be a 3-manifold and let $b_0(M)$ and $b_2(M)$ respectively be its zeroth and second $\mathbb{Z}_2$-Betti number.*

(1) For any odd integer $r \geqslant 3$,

$$TV_r(M) = TV_3(M) \cdot TV'_r(M).$$

(2) (Turaev-Viro [33]). If $\partial M = \emptyset$ and $A = e^{\frac{\pi i}{3}}$, then

$$TV_3(M) = 2^{b_2(M) - b_0(M)}.$$

(3) If M is connected, $\partial M \neq \emptyset$ and $A = e^{\frac{\pi i}{3}}$, then

$$TV_3(M) = 2^{b_2(M)}.$$

In particular, $TV_3(M)$ is nonzero.

We postpone the proof of Theorem 2.9 to Appendix A to avoid unnecessary distractions.

3 The colored Jones sum formula for Turaev-Viro invariants

In this Section, we first establish a relationship between the $SO(3)$ Turaev-Viro invariants of a 3-manifold with boundary to the $SO(3)$ Reshetikhin-Turaev invariants of its double, following the argument of [1]. See Theorem 3.1. Then we prove Theorem 1.1 using the TQFT properties of the Reshetikhin-Turaev invariants established in [2, 4].

3.1 Relationship between invariants

The relationship between Turaev-Viro and Witten-Reshetikhin-Turaev invariants was studied by Turaev-Walker [32] and Roberts [29] for closed 3-manifolds and by Beneditti and Petronio [1] for 3-manifolds with boundary. For an oriented 3-manifold M with boundary, denote M with the opposite orientation by $-M$, and let $D(M)$ denote the double of M, i.e.,

$$D(M) = M \bigcup_{\partial M} (-M).$$

We will need the following theorem of Benedetti and Petronio [1]. In fact [1] only treats the case of $A = e^{\frac{\pi i}{2r}}$, but, as we will explain below, the proof for other cases is similar.

Theorem 3.1. *Let M be a 3-manifold with boundary. Then*

$$TV_r(M) = \eta_r^{-\chi(M)} \langle D(M) \rangle_r$$

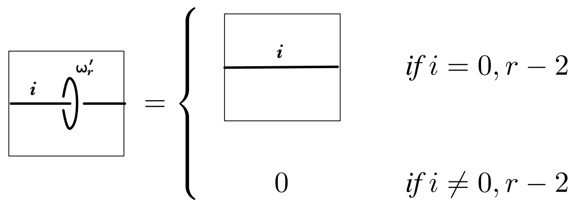
for any integer r, and

$$TV'_r(M) = (\eta'_r)^{-\chi(M)} \langle D(M) \rangle'_r$$

for any odd r, where $\chi(M)$ is the Euler characteristic of M.

We refer to [1] and [29] for the $SU(2)$ (r being any integer) case, and for the reader's convenience include a sketch of the proof here for the $SO(3)$ (r being odd) case. The main difference for the $SO(3)$ case comes from to the following lemma due to Lickorish.

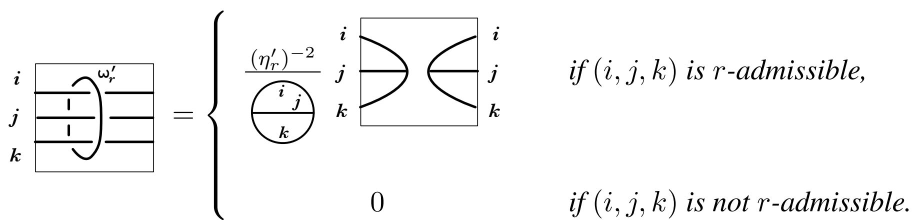
Lemma 3.2 ([18, Lemma 6])**.** *Let $r \geqslant 3$ be an odd integer and let A be a primitive $2r$-th root of unity. Then*

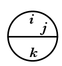


I.e., the element of the i-th Temperley-Lieb algebra obtained by circling the i-th Jones-Wenzl idempotent f_i by the Kirby coloring ω'_r equals f_i when $i = 0$ or $r - 2$, and equals 0 otherwise.

As a consequence, the usual fusion rule [19] should be modified to the following

Lemma 3.3 (Fusion Rule)**.** *Let $r \geqslant 3$ be an odd integer. Then for a triple (i, j, k) of elements of I'_r,*



Here the integers i, j and k being even is crucial, since it rules out the possibility that $i+j+k = r-2$, which by Lemma 3.2 could be troublesome. This is the reason that we consider the invariant $TV'_r(M)$ instead of $TV_r(M)$. (The factor  in the formula above is also denoted by $\theta(i, j, k)$ in [1] and [29].)

Sketch of proof of Theorem 3.1. Following [1], we extend the "chain-mail" invariant of Roberts [29] to M with non-empty boundary using a handle decomposition without 3-handles. For such a handle decomposition, let d_0, d_1 and d_2 respectively be the number of 0-, 1- and 2-handles. Let ϵ_i be the attaching curves of the 2-handles and let δ_j be the meridians of the 1-handles. Thicken the curves to bands parallel to the surface of the 1-skeleton H and push the ϵ-bands slightly into H. Embed H arbitrarily into S^3 and color each of the image of the ϵ- and δ-bands by $\eta'_r\omega'_r$ to get an element in S_M in $K_A(S^3)$. Then the chain-mail invariant of M is defined by

$$CM_r(M) = (\eta'_r)^{d_0}\langle S_M\rangle.$$

Recall here $\langle\ \rangle$ is the Kauffman bracket. It is proved in [1, 29] that $CM_r(M)$ is independent of the choice of the handle decomposition and the embedding, hence defines an invariant of M.

On the one hand, if the handle decomposition is obtained by the dual of an ideal triangulation $\mathcal{T}$ of M, namely the 2-handles come from a tubular neighborhood of the edges of $\mathcal{T}$, the 1-handles come from a tubular neighborhood of the faces of $\mathcal{T}$ and the 0-handles come from the complement of the 1- and 2-handles. Since each face has three edges, each δ-band encloses exactly three ϵ-bands (see [29, Figure 11]). By relation (2.2), every $\eta'_r\omega'_r$ on the ϵ-band can be written as

$$\eta'_r\omega'_r = \eta'_r \sum_{i=0}^{\frac{r-1}{2}-1} \langle e_i\rangle e_i = \eta'_r \sum_{i=0}^{\frac{r-1}{2}-1} \langle e_{2i}\rangle e_{2i}.$$

Next we apply Lemma 3.3 to each δ-band. In this process the four δ-bands corresponding to each tetrahedron of $\mathcal{T}$ give rise to a tetrahedral network (see also [29, Figure 12]). Then by Remark 2.5 and equations preceding it, we may rewrite $CM_r(M)$ in terms of trihedral and tetrahedral coefficients to obtain

$$CM_r(M) = (\eta'_r)^{d_0-d_1+d_2} \sum_{c\in A'_r} \frac{\prod_{e\in E} |e|^c_r \prod_{\Delta\in T} |\Delta|^c_r}{\prod_{f\in E} |f|^c_r} = (\eta'_r)^{\chi(M)} TV'_r(M).$$

On the other hand, if the handle decomposition is standard, namely H is a standard handlebody in S^3 with exactly one 0-handle, then the ϵ- and the δ-bands gives a surgery diagram L of $D(M)$. The way to see it is as follows. Consider the 4-manifold W_1 obtained by attaching 1-handles along the δ-bands (see Kirby [16]) and 2-handles along the ϵ-bands. Then W_1 is homeomorphic to $M\times I$ and $\partial W_1 = M\times\{0\}\cup \partial M\times I\cup(-M)\times\{1\} = D(M)$. Now if W_2 is the 4-manifold obtained by attaching 2-handles both the ϵ- and the δ-bands, i.e. L, then ∂W_2 is the 3-manifold represented by the framed link L. Then due to the fact that $\partial W_1 = \partial W_2$ and Definition 2.1, we have

$$CM_r(M) = \eta'_r\langle \eta'_r\omega'_r,\dots,\eta'\omega'\rangle_L = (\eta'_r)^{1+n(L)}\langle\omega'_r,\dots,\omega'_r\rangle_L = \langle D(M)\rangle'_r(\kappa'_r)^{\sigma(L)}.$$

We are left to show that $\sigma(L)=0$. It follows from that the linking matrix of L has the form

$$LK(L) = \begin{bmatrix} 0 & A \\ A^T & 0\end{bmatrix},$$

where the blocks come from grouping the ϵ- and the δ-bands together and $A_{ij} = LK(\epsilon_i,\delta_j)$. Then, for any eigen-vecor $v=(v_1,v_2)$ with eigenvalue λ, the vector $v'=(-v_1,v_2)$ is an eigen-vector of eigenvalue $-\lambda$. □

Remark 3.4. Theorems 3.1 and 2.9 together with the main result of [29] imply that if Conjecture 1.4 holds for M with totally geodesic boundary, then it holds for $D(M)$.

3.2 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. For the convenience of the reader we restate the theorem.

Theorem 1.1. *Let L be a link in S^3 with n components.*

(1) For an integer $r\geqslant 3$ and a primitive $4r$-th root of unity A, we have

$$TV_r(S^3\setminus L,q) = \eta_r^2\sum_{1\leqslant \mathbf{i}\leqslant r-1}|J_{L,\mathbf{i}}(t)|^2.$$

(2) For an odd integer $r=2m+1\geqslant 3$ and a primitive $2r$-th root of unity A, we have

$$TV_r(S^3\setminus L,q) = 2^{n-1}(\eta'_r)^2\sum_{1\leqslant \mathbf{i}\leqslant m}|J_{L,\mathbf{i}}(t)|^2.$$

Here, in both cases we have $t=q^2=A^4$.

Proof. We first consider the case that $r=2m+1$ is odd. For a framed link L in S^3 with n components, we let $M=S^3\setminus L$. Since by Theorem 2.9, $TV_r(M)=2^{n-1}TV'_r(M)$, we will work from now on with $TV'_r(M)$.

Since the Euler characteristic of M is zero, by Theorem 3.1, we obtain

$$TV'_r(M) = \langle D(M)\rangle'_r = \langle Z'_r(M), Z'_r(M)\rangle, \tag{3.1}$$

where $Z_r(M)$ is a vector in $Z_r(T^2)^{\otimes n}$. Let $\{e_i\}_{i=0,\dots,m-1}$ be the basis of $Z'_r(T^2)$ described in Theorem 2.4 (2). Then the vector space $Z_r(T^2)^{\otimes n}$ has a Hermitian basis given by $\{e_{\mathbf{i}} = e_{i_1} \otimes e_{i_2} \dots e_{i_n}\}$ for all $\mathbf{i} = (i_1, i_2, \dots, i_n)$ with $0 \leqslant \mathbf{i} \leqslant m-1$.

We write $\langle e_{\mathbf{i}}\rangle_L$ for the multi-bracket $\langle e_{i_1}, e_{i_2}, \dots, e_{i_n}\rangle_L$. Then by relation (2.1), to establish the desired formula in terms of the colored Jones polynomials, it is suffices to show the following

$$TV'_r(M) = (\eta'_r)^2 \sum_{0 \leqslant \mathbf{i} \leqslant m-1} |\langle e_{\mathbf{i}}\rangle_L|^2.$$

By writing

$$Z'_r(M) = \sum_{0 \leqslant \mathbf{i} \leqslant m-1} \lambda_{\mathbf{i}} e_{\mathbf{i}}$$

and equation (3.1), we have that

$$TV'_r(M) = \sum_{0 \leqslant \mathbf{i} \leqslant m-1} |\lambda_{\mathbf{i}}|^2.$$

The computation of the coefficients $\lambda_{\mathbf{i}}$ of $Z_r(M)$ relies on the TQFT properties of the invariants [4]. (Also compare with the argument in [5, Section 4.2]). Since $\{e_{\mathbf{i}}\}$ is a Hermitian basis of $Z_r(T^2)^{\otimes n}$, we have

$$\lambda_{\mathbf{i}} = \langle Z'_r(M), e_{\mathbf{i}}\rangle.$$

A tubular neighborhood N_L of L is a disjoint union of solid tori $\coprod_{k=1}^{n} T_k$. We let $L(e_{\mathbf{i}})$ be the element of $K_A(N_L)$ obtained by cabling the component of the L in T_k the i_k-th Chebyshev polynomial e_{i_k}. Then in $Z'_r(T)^{\otimes n}$, we have

$$e_{\mathbf{i}} = Z'_r(N_L, L(e_{\mathbf{i}})).$$

Now by Theorem 2.3 (4), since $S^3 = M \cup (-N_L)$, we have

$$\langle Z'_r(M), e_{\mathbf{i}}\rangle = \langle Z'_r(M), Z'_r(N_L, L(e_{\mathbf{i}}))\rangle = \langle M \cup (-N_L), L(e_{\mathbf{i}}))\rangle'_r = \langle S^3, L(e_{\mathbf{i}})\rangle'_r.$$

Finally, by Remark 2.2 (2), we have

$$\langle S^3, L(e_{\mathbf{i}})\rangle'_r = \eta'_r \langle e_{\mathbf{i}}\rangle_L.$$

Therefore, we have

$$\lambda_{\mathbf{i}} = \eta'_r \langle e_{\mathbf{i}}\rangle_L,$$

which finishes the proof in the case of $r = 2m+1$.

The argument of the remaining case is very similar. By Theorem 3.1, we obtain

$$TV_r(M) = \langle D(M)\rangle_r = \langle Z_r(M), Z_r(M)\rangle.$$

Working with the Hermitian basis $\{e_i\}_{i=0,\dots,r-2}$ of $Z_{2r}(T^2)$ given in Theorem 2.4 (1), we have

$$TV_r(M) = \sum_{0 \leqslant \mathbf{i} \leqslant r-2} |\lambda_{\mathbf{i}}|^2,$$

where $\lambda_{\mathbf{i}} = \langle Z_r(M), e_{\mathbf{i}}\rangle$ and $e_{\mathbf{i}} = Z_r(N_L, L(e_{\mathbf{i}}))$. Now by Theorem 2.3 (4) and Remark 2.2, one sees

$$\lambda_{\mathbf{i}} = \eta_r \langle e_{\mathbf{i}}\rangle_L,$$

which finishes the proof. □

4 Applications to Conjecture 1.4

In this section we use Theorem 1.1 to determine the asymptotic behavior of the Turaev-Viro invariants for some hyperbolic knot and link complements. In particular, we verify Conjecture 1.4 for the complement of the Figure-eight knot and the Borromean rings. To the best of our knowledge these are the first calculations of this kind.

4.1 The Figure-eight complement

The following theorem verifies Conjecture 1.4 for the Figure-eight knot.

Theorem 4.1. *Let K be the Figure-eight knot and let M be the complement of K in S^3. Then*

$$\lim_{r\to+\infty}\frac{2\pi}{r}\log TV_r(M,e^{\frac{2\pi i}{r}})=\lim_{m\to+\infty}\frac{4\pi}{2m+1}\log|J_{K,m}(e^{\frac{4\pi i}{2m+1}})|=Vol(M),$$

where $r=2m+1$ runs over all odd integers.

Proof. By Theorem 1.1, we have for odd $r=2m+1$ that

$$TV_r(S^3\smallsetminus K,e^{\frac{2\pi i}{r}})=(\eta_r')^2\sum_{i=1}^{m}|J_i(K,t)|^2,$$

where $t=q^2=e^{\frac{4\pi i}{r}}$. Notice that $(\eta_r')^2$ grows only polynomially in r.

By Habiro and Le's formula [11], we have

$$J_{K,i}(t)=1+\sum_{j=1}^{i-1}\prod_{k=1}^{j}\Big(t^{\frac{i-k}{2}}-t^{-\frac{i-k}{2}}\Big)\Big(t^{\frac{i+k}{2}}-t^{-\frac{i+k}{2}}\Big),$$

where $t=A^4=e^{\frac{4\pi i}{r}}$.

For each i define the function $g_i(j)$ by

$$\begin{aligned}g_i(j)&=\prod_{k=1}^{j}\Big|\Big(t^{\frac{i-k}{2}}-t^{-\frac{i-k}{2}}\Big)\Big(t^{\frac{i+k}{2}}-t^{-\frac{i+k}{2}}\Big)\Big|\\&=\prod_{k=1}^{j}4\Big|\sin\frac{2\pi(i-k)}{r}\Big|\Big|\sin\frac{2\pi(i+k)}{r}\Big|.\end{aligned}$$

Then

$$|J_{K,i}(t)|\leqslant 1+\sum_{j=1}^{i-1}g_i(j).$$

Now let i be such that $\frac{i}{r}\to a\in[0,\frac{1}{2}]$ as $r\to\infty$. For each i, let $j_i\in\{1,\dots,i-1\}$ such that $g_i(j_i)$ achieves the maximum. We have that $\frac{j_i}{r}$ converges to some $j_a\in(0,1/2)$ which varies continuously in a when a is close to $\frac{1}{2}$. Then

$$\lim_{r\to\infty}\frac{1}{r}\log|J_{K,i}|\leqslant\lim_{r\to\infty}\frac{1}{r}\log\Big(1+\sum_{j=1}^{i-1}g_i(j)\Big)=\lim_{r\to\infty}\frac{1}{r}\log\big(g_i(j_i)\big),$$

where the last term equals

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r}\Big(\sum_{k=1}^{j_i}\log\Big|2\sin\frac{2\pi(i-k)}{r}\Big|+\sum_{k=1}^{j_i}\log\Big|2\sin\frac{2\pi(i+k)}{r}\Big|\Big)\\
=&\frac{1}{2\pi}\int_0^{j_a\pi}\log\Big(2|\sin\big(2\pi a-t\big)|\Big)dt+\frac{1}{2\pi}\int_0^{j_a\pi}\log\Big(2|\sin\big(2\pi a+t\big)|\Big)dt\\
=&-\frac{1}{2\pi}\Big(\Lambda\big(2\pi(j_a-a)\big)+\Lambda\big(2\pi a\big)\Big)-\frac{1}{2\pi}\Big(\Lambda\big(2\pi(j_a+a)\big)-\Lambda\big(2\pi a\big)\Big)\\
=&-\frac{1}{2\pi}\Big(\Lambda\big(2\pi(j_a-a)\big)+\Lambda\big(2\pi(j_a+a)\big)\Big).
\end{aligned}$$

Since $\Lambda(x)$ is an odd function and achieves the maximum at $\frac{\pi}{6}$, the last term above is less than or equal to

$$\frac{\Lambda(\frac{\pi}{6})}{\pi}=\frac{3\Lambda(\frac{\pi}{3})}{2\pi}=\frac{Vol(S^3\smallsetminus K)}{4\pi}.$$

We also notice that for $i=m$, $\frac{i}{r}=\frac{m}{2m+1}\to\frac{1}{2}$, $j_{\frac{1}{2}}=\frac{5}{12}$ and all the inequalities above become equalities. Therefore, the term $|J_{K,m}(t)|^2$ grows the fastest, and

$$\lim_{r\to+\infty}\frac{2\pi}{r}\log TV_r(S^3\smallsetminus K,A^2)=\lim_{r\to+\infty}\frac{2\pi}{r}\log|J_{K,m}(t)|^2=Vol(S^3\smallsetminus K).$$

□

4.2 The Borromean rings complement

In this subsection we prove the following theorem that verifies Conjecture 1.4 for the 3-component Borromean rings.

Theorem 4.2. *Let L be the 3-component the Borromean rings, and let M be the complement of L in S^3. Then*

$$\lim_{r\to+\infty}\frac{2\pi}{r}\log TV_r(M,e^{\frac{2\pi i}{r}})=\lim_{m\to+\infty}\frac{4\pi}{2m+1}\log|J_{L,m}(e^{\frac{4\pi i}{2m+1}})|=Vol(M),$$

where $r=2m+1$ runs over all odd integers. Here, $J_{L,m}$ denotes the colored Jones polynomial where all the components of L are colored by m.

The proof relies on the following formula for the colored Jones polynomials of the Borromean rings given by Habiro [11, 12]. Let L be the Borromean rings and k, l and n be non-negative integers. Then

$$J_{L,(k,l,n)}(t)=\sum_{j=0}^{\min(k,l,n)-1}(-1)^j\frac{[k+j]![l+j]![n+j]!}{[k-j-1]![l-j-1]![n-j-1]!}\left(\frac{[j]!}{[2j+1]!}\right)^2. \tag{4.1}$$

Recall that in this formula $[n]=\frac{t^{n/2}-t^{-n/2}}{t^{1/2}-t^{-1/2}}$. From now on we specialize at $t=e^{\frac{4\pi i}{r}}$ where $r=2m+1$. We have that

$$[n]=\frac{2\sin(\frac{2n\pi}{r})}{2\sin(\frac{2\pi}{r})}=\frac{\{n\}}{\{1\}},$$

where we write $\{j\}=2\sin(\frac{2j\pi}{r})$. Then we can rewrite formula (4.1) as

$$J_{L,(k,l,n)}(e^{\frac{4i\pi}{r}})=\sum_{j=0}^{\min(k,l,n)-1}(-1)^j\frac{1}{\{1\}}\frac{\{k+j\}!\{l+j\}!\{n+j\}!}{\{k-j-1\}!\{l-j-1\}!\{n-j-1\}!}\left(\frac{\{j\}!}{\{2j+1\}!}\right)^2.$$

Next we establish three lemmas needed for the proof of Theorem 4.2.

Lemma 4.3. *For r odd, let ev_r denote the evaluation of a Laurent polynomial at $A = e^{\frac{\pi i}{r}}$. Then for any integer j with $0 < j < r$, we have*

$$\log(|ev_r(\{\,j\,\}!)|) = -\frac{r}{2\pi}\Lambda(\frac{2j\pi}{r}) + O(\log(r)),$$

where $O(\log(r))$ is uniform.

Proof. This result is an adaptation of the result in [9] for r odd. We recall the argument for the sake of completeness. By the Euler-Mac Laurin summation formula, for any twice differentiable function f on $[a, b]$ where a and b are integer, we have

$$\sum_{k=a}^{b} f(k) = \int_a^b f(t)dt + \frac{1}{2}f(a) + \frac{1}{2}f(b) + R(a, b, f),$$

where

$$|R(a, b, f)| \leqslant \frac{3}{24}\int_a^b |f''(t)|dt.$$

Applying this to

$$\log(|ev_r(\{\,j\,\}!)|) = \sum_{k=1}^{j} \log\left(2|\sin(\frac{2k\pi}{r})|\right),$$

we get

$$\begin{array}{rcl} (|ev_r(\{\,j\,\}!)|) & = & \int_1^j \log(2|\sin(\frac{2t\pi}{r}) + \frac{1}{2}(f(1) + f(j)) + R(\frac{2\pi}{r}, \frac{2j\pi}{r}, f) \\ & = & \frac{r}{2\pi}\int_{\frac{2\pi}{r}}^{\frac{2j\pi}{r}} \log(2|\sin(\frac{2t\pi}{r}) + \frac{1}{2}(f(1) + f(j)) + R(\frac{2\pi}{r}, \frac{2j\pi}{r}, f) \\ & = & \frac{r}{2\pi}(-\Lambda(\frac{2j\pi}{r}) + \Lambda(\frac{2\pi}{r})) + \frac{1}{2}(f(1) + f(j)) + R(\frac{2\pi}{r}, \frac{2j\pi}{r}, f), \end{array}$$

where $f(t) = \log(2|\sin(\frac{2t\pi}{r})|)$.

As $|r\Lambda(\frac{2\pi}{r})| \leqslant C\log(r)$ and $|f(1) + f(j)| \leqslant C'\log(r)$ for constants C and C' independent of j, and as

$$R(1, j, f) = \int_1^j |f''(t)|dt = \int_1^j \frac{4\pi^2}{r^2}\frac{1}{\sin(\frac{2\pi t}{r})^2} = \frac{2\pi}{r}\left(\cot(\frac{2j\pi}{r}) - \cot(\frac{2\pi}{r})\right) = O(1),$$

we get

$$\log(|ev_r(\{\,j\,\}!)|) = -\frac{r}{2\pi}\Lambda(\frac{2j\pi}{r}) + O(\log(r))$$

as claimed. $\square$

Lemma 4.3 allows us to get an estimation of terms that appear in Habiro's sum for the multi-bracket of Borromean rings. We find that

$$\log\left|\frac{1}{\{1\}}\frac{\{k+j\}!\{l+j\}!\{n+j\}!}{\{k-i-1\}!\{l-i-1\}!\{n-i-1\}!}\left(\frac{\{i\}!}{\{2i+1\}!}\right)^2\right|$$
$$= -\frac{r}{2\pi}(f(\alpha,\theta) + f(\beta,\theta) + f(\gamma,\theta)) + O(\log(r)),$$

where $\alpha = \frac{2k\pi}{r}$, $\beta = \frac{2l\pi}{r}$, $\gamma = \frac{2n\pi}{r}$ and $\theta = \frac{2j\pi}{r}$, and

$$f(\alpha,\theta) = \Lambda(\alpha+\theta) - \Lambda(\alpha-\theta) + \frac{2}{3}\Lambda(\theta) - \frac{2}{3}\Lambda(2\theta).$$

Lemma 4.4. *The minimum of the function $f(\alpha,\theta)$ is $-\frac{8}{3}\Lambda(\frac{\pi}{4}) = -\frac{v_8}{3}$. This minimum is attained for $\alpha = 0$ modulo π and $\theta = \frac{3\pi}{4}$ modulo π.*

Proof. The critical points of f are given by the conditions

$$\begin{cases} \Lambda'(\alpha+\theta) - \Lambda'(\alpha-\theta) = 0 \\ \Lambda'(\alpha+\theta) + \Lambda'(\alpha-\theta) + \frac{2}{3}\Lambda'(\theta) - \frac{4}{3}\Lambda'(2\theta) = 0. \end{cases}$$

As $\Lambda'(x) = 2\log|\sin(x)|$, the first condition is equivalent to $\alpha+\theta = \pm\alpha - \theta \mod \pi$. Thus, either $\theta = 0 \mod \frac{\pi}{2}$ in which case $f(\alpha,\theta) = 0$, or $\alpha = 0$ or $\frac{\pi}{2} \mod \pi$.

In the second case, as the Lobachevski function has the symmetries $\Lambda(-\theta) = -\Lambda(\theta)$ and $\Lambda(\theta+\frac{\pi}{2}) = \frac{1}{2}\Lambda(2\theta) - \Lambda(\theta)$, we get

$$f(0,\theta) = \frac{8}{3}\Lambda(\theta) - \frac{2}{3}\Lambda(2\theta),$$

and

$$f(\frac{\pi}{2},\theta) = \frac{1}{3}\Lambda(2\theta) - \frac{4}{3}\Lambda(\theta).$$

We get critical points when $2\Lambda'(\theta) = \Lambda(2\theta)$ which is equivalent to $(2\sin(\theta))^2 = 2|\sin(2\theta)|$. This happens only for $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4} \mod \pi$ and the minimum value is $-\frac{8}{3}\Lambda(\frac{\pi}{4})$, which is obtained only for $\alpha = 0 \mod \pi$ and $\theta = \frac{3\pi}{4} \mod \pi$ only. □

Lemma 4.5. *If $r = 2m+1$, we have that*

$$\log(|J_{L,(m,m,m)}(e^{\frac{4i\pi}{r}})|) = \frac{r}{2\pi}v_8 + O(\log(r)).$$

Proof. Again, the argument is very similar to the argument of the usual volume conjecture for the Borromean ring in Theorem A.1 of [9]. We remark that quantum integer $\{n\}$ admit the symmetry that

$$\{m+1+i\} = -\{m-i\}$$

for any integer i.

Now, for $k = l = n = m$, Habiro's formula for the colored Jones polynomial turns into

$$\begin{aligned} J_{L,(m,m,m)}(t) &= \sum_{j=0}^{m-1}(-1)^j\frac{\{m\}^3}{\{1\}}\left(\prod_{k=1}^{j}\{m+k\}\{m-k\}\right)^3\left(\frac{\{j\}!}{\{2j+1\}!}\right)^2 \\ &= \sum_{j=0}^{m-1}\frac{\{m\}^3\{m+j+1\}}{\{1\}\{m+1\}}\left(\prod_{k=1}^{j}\{m+k\}\right)^6\left(\frac{\{j\}!}{\{2j+1\}!}\right)^2. \end{aligned}$$

Note that as $\{n\} = \sin(\frac{2\pi n}{2m+1}) < 0$ for $n \in \{m+1, m+2, \ldots, 2m\}$, the factor $\{m+j+1\}$ will always be negative for $0 \leqslant m-1$. Thus all terms in the sum have the same sign. Moreover, there is only a polynomial in r number of terms in the sum as $m = \frac{r-1}{2}$. Therefore, $\log(|J_{L,(m,m,m)}|)$ is up to $O(\log(r))$ equal to the logarithm of the biggest term. But the term $j = \lfloor\frac{3r}{8}\rfloor$ corresponds to $\alpha = \frac{2(m-1)\pi}{r} = 0 + O(\frac{1}{r}) \mod \pi$ and $\theta = \frac{2j\pi}{r} = \frac{3\pi}{4} + O(\frac{1}{r}) \mod \pi$, so

$$\log\left|\frac{\{m\}^3}{\{1\}}\left(\prod_{k=1}^{m-1}\{m+k\}\{m-k\}\right)^3\left(\frac{\{m-1\}!}{\{2m-1\}!}\right)^2\right| = \frac{r}{2\pi}v_8 + O(\log(r)),$$

and

$$\frac{2\pi}{r}\log|J_{L,(m,m,m)}| = v_8 + O(\frac{\log(r)}{r}).$$

□

Proof of Theorem 4.2. Theorem 1.1, we have

$$TV_r'(S^3\smallsetminus L, e^{\frac{2\pi i}{r}}) = (\eta_r')^2 \sum_{1\leqslant k,l,n\leqslant m} |J_{L,(k,l,n)})(e^{\frac{4i\pi}{r}})|^2.$$

This is a sum of $m^3 = \left(\frac{r-1}{2}\right)^3$ terms, the logarithm of all of which are less than $\frac{r}{2\pi}(2v_8) + O(\log(r))$ by Lemma 4.4. Also, the term $|J_{L,(m,m,m)}(e^{\frac{4i\pi}{r}})|^2$ has logarithm $\frac{r}{2\pi}(2v_8) + O(\log(r))$. Thus we have

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r'(S^3\smallsetminus L), e^{\frac{2\pi i}{r}}) = 2v_8 = \mathrm{Vol}(S^3\smallsetminus L).$$

$\square$

Finally we note that Theorem 1.6 stated in the introduction follows by Theorems 4.1 and 4.2.

5 Turaev-Viro Invariants and simplicial volume

Given a link L in S^3 one can consider the toroidal decomposition of its complement. Recall that the simplicial volume (or Gromov norm) of L, denoted by $||L||$, is the sum of the volumes of the hyperbolic pieces of the decomposition, divided by v_3 the volume of the regular ideal tetrahedron in the hyperbolic space. In particular, if the toroidal decomposition has no hyperbolic pieces, then we have $||L|| = 0$. Soma [30] has shown that the simplicial volume is additive under split union and connected sums of links. That is

$$||L_1 \sqcup L_2|| = ||L_1 \# L_2|| = ||L_1|| + ||L_2||.$$

We note that the connected sum for links is not uniquely defined, it depends on the components of links being connected.

Conjecture 5.1. *For every link $L \subset S^3$, we have*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(S^3\smallsetminus L, e^{\frac{2\pi i}{r}})) = v_3||L||,$$

where r runs over all odd integers.

Theorem 1.1 suggests that the Turaev-Viro invariants are a more natural object to study for the volume conjecture for links. As remarked in [25] that all the Kashaev invariants of a split link are zero. As a result, the original volume conjecture [25] is not true for split links. On the other hand, Corollary 1.2 implies that $TV_r'(S^3\smallsetminus L, q) \neq 0$ for any $r \geqslant 3$ and any primitive root of unity $q = A^2$.

Define the double of a knot complement to be the double of the complement of a tubular neighborhood of the knot. Then Theorem 3.1 and the main result of [29] implies that if Conjecture 5.1 holds for a link, then it holds for the double of its complement. In particular, as a consequence of Theorem 1.6, we have

Corollary 5.2. *Conjecture 5.1 is true for the double of the Figure-eight and the Borromean rings complement.*

Since colored Jones polynomials are multiplicative under split union of links, Theorem 1.1 also implies that $TV_r'(S^3\smallsetminus L, q)$ is up to a factor multiplicative under split union.

Corollary 5.3. *For any odd integer $r \geqslant 3$ and $q = A^2$ for a primitive $2r$-th root of unit A,*

$$TV_r'(S^3\smallsetminus(L_1 \sqcup L_2), q) = (\eta_r')^{-1} TV_r'(S^3\smallsetminus L_1, q) \cdot TV_r'(S^3\smallsetminus L_2, q).$$

The additivity of simplicial volume implies that Conjecture 5.1 is true for L_1 and L_2, then it is true for the split union $L_1 \sqcup L_2$.

We next discuss the behavior of the Turaev-Viro invariants under taking connected sums of links. With our normalization of the colored Jones polynomials, we have for a connected sum of two like $L_1 \# L_2$ that

$$J_{L_1\#L_2,\mathbf{i}}(t) = [i] J_{L_1,\mathbf{i}_1}(t) \cdot J_{L_2,\mathbf{i}_2}(t),$$

where $\mathbf{i}_1$ and $\mathbf{i}_2$ are respectively the restriction of $\mathbf{i}$ to L_1 and L_2, and i is the component of $\mathbf{i}$ corresponding to the component of $L_1 \# L_2$ coming from the connected sum. This implies the following

Corollary 5.4. *For any odd integer $r \geqslant 3$, $q = A^2$ and $t = A^4$ for a primitive $2r$-th root of unit A,*

$$TV'_r(S^3 \smallsetminus L_1 \# L_2, q) = (\eta'_r)^2 \sum_{1 \leqslant \mathbf{i} \leqslant m} [i]^2 |J_{L_1,\mathbf{i}_1}(t)|^2 |J_{L_2,\mathbf{i}_2}(t)|^2,$$

where $\mathbf{i}_1$ and $\mathbf{i}_2$ are respectively the restriction of $\mathbf{i}$ to L_1 and L_2, and i is the component of $\mathbf{i}$ corresponding to the component of $L_1 \# L_2$ coming from the connected sum.

In the rest of this section, we focus on the value $q = e^{\frac{2i\pi}{r}}$ for odd $r = 2m+1$. Notice in this case that the quantum integers $[i]$ for $1 \leqslant i \leqslant m$ are non-zero and their log are of order $O(\log r)$. Corollary 5.4 implies that

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{2\pi}{r} \log TV'_r(S^3 \smallsetminus L_1 \# L_2, q) \\
&\qquad\qquad \leqslant \limsup_{r\to\infty} \frac{2\pi}{r} \log TV'_r(S^3 \smallsetminus L_1, q) + \limsup_{r\to\infty} \frac{2\pi}{r} \log TV'_r(S^3 \smallsetminus L_2, q).
\end{aligned}$$

Moreover if we assume a positive answer to Question 1.7 for L_1 and L_2, then the term $|J_{L_1\#L_2,m}(t)|^2$ of the sum for $L_1 \# L_2$ satisfies

$$\lim_{r\to\infty} \frac{2\pi}{r} \log |J_{L_1\#L_2,m}(t)|^2 = \mathrm{Vol}(S^3 \smallsetminus L_1 \# L_2).$$

It follows that if the answer to Question 1.7 is positive, and Conjecture 5.1 is true for links L_1 and L_2, then Conjecture 5.1 is true for their connected sum. In particular, Theorem 1.6 implies the following

Corollary 5.5. *Conjecture 5.1 is true for any link obtained by connected sum of the Figure-eight and the Borromean rings.*

We finish the section with the proof of Theorem 1.8, verifying Conjecture 5.1 for knots of simplicial volume zero.

Theorem 1.8. *Let $K \subset S^3$ be knot with simplicial volume zero. Then, we have*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log(TV_r(S^3 \smallsetminus K, e^{\frac{2\pi i}{r}})) = ||K|| = 0,$$

where r runs over all odd integers.

Proof. By part (2) of Theorem 1.1 we have

$$TV_r(S^3 \smallsetminus K, e^{\frac{2i\pi}{r}}) = (\eta'_r)^2 \sum_{1 \leqslant i \leqslant m} |J_{L,i}(e^{\frac{4i\pi}{r}})|^2. \tag{5.1}$$

Since $J_{K,1}(t) = 1$, we have $TV_r(S^3\setminus K) \geqslant \eta_r'^2 > 0$ for any knot K. Thus for $r >> 0$ the sum of the values of the colored Jones polynomials in (5.1) is larger or equal to 1. On the other hand, we have $\eta_r' \neq 0$ and $\dfrac{\log(|\eta_r'|^2)}{r} \to 0$ as $r \to \infty$. Therefore,

$$\liminf_{r\to\infty} \frac{\log|TV_r(S^3\setminus K)|}{r} \geqslant 0.$$

Now we only need to prove for simplicial volume zero knots that

$$\limsup_{r\to\infty} \frac{\log|TV_r(S^3\setminus K)|}{r} \leqslant 0.$$

By Theorem 1.1, (2) again, it suffices to prove that the L^1-norm $||J_{K,i}(t)||$ of the colored Jones polynomials of any knot K of simplicial volume zero is bounded by a polynomial in i. By Gordon [10], the set of knots of simplicial volume zero is generated by the torus knots, and is closed under taking connected sums and cablings. Therefore, it suffices to prove that the set of knots whose colored Jones polynomials have L^1-norm growing at most polynomially contains the torus knots, and is closed under taking connected sums and cablings.

From Morton's formula [21], for the torus knot $T_{p,q}$, we have

$$J_{T_{p,q},i}(t) = t^{pq(1-i^2)} \sum_{|k|=-\frac{i-1}{2}}^{\frac{i-1}{2}} \frac{t^{4pqk^2-4(p+q)k+2} - t^{4pqk^2-4(p-q)k-2}}{t^2 - t^{-2}}.$$

Each fraction in the summation can be simplified to a geometric sum of powers of t^2, and hence has L^1-norm less than $2qi+1$. From this we have $||J_{T_{p,q},i}(t)|| = O(i^2)$.

For a connected sum of knots, we recall that the L^1-norm of a Laurent polynomial is

$$||\sum_{d\in\mathbb{Z}} a_d t^d|| = \sum_{d\in\mathbb{Z}} |a_d|.$$

For a Laurent polynomial $R(t) = \sum\limits_{f\in\mathbb{Z}} c_f t^f$, we let

$$deg(R(t)) = max(\{d\,/c_d \neq 0\}) - min(\{d\,/c_d \neq 0\}).$$

Then for two Laurent polynomials $P(t) = \sum\limits_{d\in\mathbb{Z}} a_d t^d$ and $Q(t) = \sum\limits_{e\in\mathbb{Z}} b_e t^e$, we have

$$\begin{aligned} ||PQ|| = ||\left(\sum_{d\in\mathbb{Z}} a_d t^d\right)\left(\sum_{d\in\mathbb{Z}} b_d t^d\right)|| &\leqslant ||\sum_{f\in\mathbb{Z}}\left(\sum_{d+e=f} a_d b_e\right) t^f|| \\ &\leqslant \deg(PQ) \sum_{d+e=f} |a_d b_e| \\ &\leqslant \deg(PQ)||P||||Q||. \end{aligned}$$

Since the L^1-norm of $[i]$ grows polynomially in i, if the L^1-norms of $J_{K_1,i}(t)$ and $J_{K_2,i}(t)$ grow polynomially, then so does that of $J_{K_1\#K_2,i}(t) = [i]J_{K_1,i}(t) \cdot J_{K_2,i}(t)$.

Finally, for the (p,q)-cabling $K_{p,q}$ of a knot K, the cabling formula [22, 34] says

$$J_{K_{p,q},i}(t) = t^{pq(i^2-1)/4} \sum_{k=-\frac{i-1}{2}}^{\frac{i-1}{2}} t^{-pk(qk+1)} J_{K,2qn+1}(t),$$

where k runs over integers if i is odd and over half-integers if i is even. It implies that if $||J_{K,i}(t)|| = O(i^d)$, then $||J_{K_{p,q},i}(t)|| = O(i^{d+1})$. $\square$

By Theorem 1.1 and the argument in the beginning of the proof of Theorem 1.8 applied to links we obtain the following.

Corollary 5.6. *For every link $L \subset S^3$, we have*

$$\liminf_{r\to\infty} \frac{\log |TV_r(S^3 \setminus L)|}{r} \geqslant 0,$$

where r runs over all odd integers.

As said earlier, there is no lower bound for the growth rate of the Kashaev invariants that holds for all links; and no such bound is known for knots as well.

A The relationship between $TV_r(M)$ and $TV'_r(M)$

The goal of this appendix is to prove Theorem 2.9. To this end, it will be convenient to modify the definition of the Turaev-Viro invariants given in Subsection 2 and use the formalism of quantum $6j$-symbols as in [33].

For $i \in I_r$, we let

$$|i| = (-1)^i [i+1],$$

for each admissible triple (i,j,k), we let

$$\big|i,j,k\big| = (-1)^{-\frac{i+j+k}{2}} \frac{[\frac{i+j-k}{2}]![\frac{j+k-i}{2}]![\frac{k+i-j}{2}]!}{[\frac{i+j+k}{2}+1]!},$$

and for each admissible 6-tupe (i,j,k,l,m,n), we let

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z [z+1]!}{[z-T_1]![z-T_2]![z-T_3]![z-T_4]![Q_1-z]![Q_2-z]![Q_3-z]!}.$$

Consider a triangulation $\mathcal{T}$ of M. Let c be an admissible coloring of $(M,\mathcal{T})$ at level r. For each edge e of $\mathcal{T}$, we let

$$|e|_c = |c(e)|,$$

for each face f with edges e_1, e_2 and e_3, we let

$$|f|_c = \big|c(e_1), c(e_2), c(e_3)\big|,$$

and for each tetrahedra Δ with edges e_{ij}, $\{i,j\} \subset \{1,\dots,4\}$, we let

$$|\Delta|_c = \begin{vmatrix} c(e_{12}) & c(e_{13}) & c(e_{23}) \\ c(e_{34}) & c(e_{24}) & c(e_{14}) \end{vmatrix}.$$

Now recall the invariants $TV_r(M)$ and $TV'_r(M)$ given in Definitions 2.7 and 2.8, respectively. Then we have the following.

Proposition A.1. *(a) For any integer $r \geqslant 3$,*

$$TV_r(M) = \eta_r^{2|V|} \sum_{c\in A_r} \prod_{e\in E} |e|_c \prod_{f\in E} |f|_c \prod_{\Delta\in T} |\Delta|_c.$$

(b) For any odd integer $r \geqslant 3$,

$$TV'_r(M) = (\eta'_r)^{2|V|} \sum_{c \in A'_r} \prod_{e \in E} |e|_c \prod_{f \in E} |f|_c \prod_{\Delta \in T} |\Delta|_c.$$

Proof. The proof is a straightforward calculation. $\square$

Next we establish four lemmas on which the proof of Theorem 2.9 will rely. We will use the notations $|i|_r$, $|i,j,k|_r$ and $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r$ respectively mean the values of $|i|$, $|i,j,k|$ and $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ at a primitive $2r$-th root of unity A.

Lemma A.2. $|0|_3 = |1|_3 = 1$, $\big|0,0,0\big|_3 = \big|1,1,0\big|_3 = 1$ *and*

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_3 = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}_3 = 1.$$

Proof. A direct calculation. $\square$

The following lemma can be considered as a Turaev-Viro setting analogue of Theorem 2.4 (3).

Lemma A.3. *For $i \in I_r$, let $i' = r - 2 - i$.*

(a) If $i \in I_r$, then $i' \in I_r$. Moreover, $|i'|_r = |i|_r$.

(b) If the triple (i,j,k) is admissible, then so is (i',j',k). Moreover,

$$\big|i',j',k\big|_r = \big|i,j,k\big|_r.$$

(c) If the 6-tuple (i,j,k,l,m,n) is admissible, then so are (i,j,k,l',m',n') and (i',j',k,l',m',n). Moreover,

$$\begin{vmatrix} i & j & k \\ l' & m' & n' \end{vmatrix}_r = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r \quad \textit{and} \quad \begin{vmatrix} i' & j' & k \\ l' & m' & n \end{vmatrix}_r = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r.$$

Proof. (a) (b) are straightforward from definition.

To see the first identity of (c), let T'_i and Q'_j be the corresponding sums for (i,j,k,l',m',n'), namely

$$T'_1 = \frac{i+j+k}{2} = T_1, \ \ T'_2 = \frac{j+l'+n'}{2} \ \text{ and } Q'_2 = \frac{i+k+l'+n'}{2}, \ \text{ etc.}$$

For the terms in the summation, let us leave T_1 alone for now, and consider the other T_i's and Q_j's. The key observation is that, if without loss of generality $Q_3 \geqslant Q_2 \geqslant Q_1 \geqslant T_4 \geqslant T_3 \geqslant T_2$, then one can easily check

(1) $Q_3 - Q_1 = T'_4 - T'_2$, $Q_2 - Q_1 = T'_4 - T'_3$, $Q_1 - T_4 = Q'_1 - T'_4$, $T_4 - T_3 = Q'_2 - Q'_1$ and $T_4 - T_2 = Q'_3 - Q'_1$, which implies

(2) $Q'_3 \geqslant Q'_2 \geqslant Q'_1 \geqslant T'_4 \geqslant T'_3 \geqslant T'_2$.

For z in between $\max\{T_1,\dots,T_4\}$ and $\min\{Q_1,Q_2,Q_3\}$, let

$$P(z)=\frac{(-1)^z[z+1]!}{[z-T_1]![z-T_2]![z-T_3]![z-T_4]![Q_1-z]![Q_2-z]![Q_3-z]!},$$

and similarly for z in between $\max\{T'_1,\dots,T'_4\}$ and $\min\{Q'_1,Q'_2,Q'_3\}$ let

$$P'(z)=\frac{(-1)^z[z+1]!}{[z-T'_1]![z-T'_2]![z-T'_3]![z-T'_4]![Q'_1-z]![Q'_2-z]![Q'_3-z]!}.$$

Then for any $a\in\{0,1,\dots,Q_1-T_4=Q'_1-T'_4\}$ one verifies by (1) above that

$$P(T_4+a)=P'(Q'_1-a). \tag{A.1}$$

There are the following three cases to consider.

Case 1. $T_1\leqslant T_4$ and $T'_1\leqslant T'_4$. In this case $T_{\text{max}}=T_4$, $Q_{\text{min}}=Q_1$, $T'_{\text{max}}=T'_4$ and $Q'_{\text{min}}=Q'_1$. By (A.1), we have

$$\sum_{z=T_4}^{Q_1}P(z)=\sum_{a=0}^{Q_1-T_4}P(T_4+a)=\sum_{a=0}^{Q'_1-T_4}P'(Q'_1-a)=\sum_{z=T'_4}^{Q'_1}P'(z).$$

Case 2. $T_1>T_4$ but $T'_1<T'_4$, or $T_1<T_4$ but $T'_1>T'_4$. But symmetry, it suffices to consider the former case. In this case $T_{\text{max}}=T_1$, $Q_{\text{min}}=Q_1$, $T'_{\text{max}}=T'_4$ and $Q'_{\text{min}}=Q'_1$, and

$$Q'_1-(r-2)=\frac{i+j-l-m}{2}=T_1-T_4.$$

As a consequence $Q'_1>r-2$. By (A.1), we have

$$\sum_{z=T_1}^{Q_1}P(z)=\sum_{a=T_1-T_4}^{Q_1-T_4}P(T_4+a)=\sum_{a=Q'_1-(r-2)}^{Q'_1-T'_4}P'(Q'_1-a)=\sum_{z=T'_4}^{r-2}P'(z)=\sum_{z=T'_4}^{Q'_1}P'(z).$$

The last equality is because that $P'(z)=0$ for $z>r-2$.

Case 3. $T_1>T_4$ and $T'_1>T'_4$. In this case $T_{\text{max}}=T_1$, $Q_{\text{min}}=Q_1$, $T'_{\text{max}}=T'_1$ and $Q'_{\text{min}}=Q'_1$. We have

$$Q'_1-(r-2)=\frac{i+j-l-m}{2}=T_1-T_4>0,$$

hence $Q_1>r-2$. Also, we have

$$Q'_1-T'_1=\frac{l'+m'-k}{2}=r-2-T_4.$$

As a consequence, $Q'_1-(r-2)=T'_1-T_4=T_1-T_4>0$, and hence $Q'_1>r-2$. By (A.1), we have

$$\sum_{z=T_1}^{Q_1}P(z)=\sum_{z=T_1}^{r-2}P(z)=\sum_{a=T_1-T_4}^{r-2-T_4}P(T_4+a)=\sum_{a=Q'_1-(r-2)}^{Q'_1-T'_1}P'(Q'_1-a)=\sum_{z=T'_1}^{r-2}P'(z)=\sum_{z=T'_1}^{Q'_1}P'(z).$$

The first and the last equality are because that $P(z)=P'(z)=0$ for $z>r-2$.

The second identity of (c) is a consequence of the first. □

As an immediate consequence of the two lemmas above, we have

Lemma A.4. *(a) For all* $i \in I_r$, $|i|_r = |0|_3|i|_r$ *and* $|i'|_r = |1|_3|i|_r$.

(b) If the triple (i,j,k) *is admissible, then*

$$\big|i,j,k\big|_r = \big|0,0,0\big|_3\big|i,j,k\big|_r \quad \textit{and} \quad \big|i',j',k\big|_r = \big|1,1,0\big|_3\big|i,j,k\big|_r.$$

(c) For all admissible 6*-tuple* (i,j,k,l,m,n),

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}_3 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r,$$

$$\begin{vmatrix} i & j & k \\ l' & m' & n' \end{vmatrix}_r = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}_3 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r,$$

$$\begin{vmatrix} i' & j' & k \\ l' & m' & n \end{vmatrix}_r = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}_3 \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_r.$$

□

Now we are ready to prove Theorem 2.9.

Proof of Theorem 2.9. For (a), we observe that there is a bijection $\phi : I_3 \times I'_r \to I_r$ defined by $\phi(0,i) = i$ and $\phi(1,i) = i'$. This induces a bijection

$$\phi : A_3 \times A'_r \to A_r.$$

Then by Proposition A.1, we have

$$\begin{aligned}
&TV_3(M) \cdot TV'_r(M) \\
&\quad = \Big(\eta_3^{2|V|} \sum_{c\in A_3} \prod_{e\in E} |e|_c \prod_{f\in F} |f|_c \prod_{\Delta\in T} |\Delta|_c\Big)\Big(\eta_r'^{2|V|} \sum_{c'\in A'_r} \prod_{e\in E} |e|_{c'} \prod_{f\in F} |f|_{c'} \prod_{\Delta\in T} |\Delta|_{c'}\Big) \\
&\quad = (\eta_3\eta'_r)^{2|V|} \sum_{(c,c')\in A_3\times A'_r} \prod_{e\in E} |e|_c|e|_{c'} \prod_{f\in F} |f|_c|f|_{c'} \prod_{\Delta\in T} |\Delta|_c|\Delta|_{c'} \\
&\quad = \eta_r^{2|V|} \sum_{\phi(c,c')\in A_r} \prod_{e\in E} |e|_{\phi(c,c')} \prod_{f\in F} |f|_{\phi(c,c')} \prod_{\Delta\in T} |\Delta|_{\phi(c,c')} \\
&\quad = TV_r(M),
\end{aligned}$$

where the third equality comes from $\eta_r = \eta_3 \cdot \eta'_r$ and Lemma A.4. This finishes the proof of part (a) of the statement of the theorem. Part (b) is given in [33, 9.3.A].

To deduce (c), by Lemma A.2 we have that

$$TV_3(M) = \sum_{c\in A_3} 1 = |A_3|.$$

Note that $c \in A_3$ if and only if $c(e_1) + c(e_2) + c(e_3)$ is even for the edges e_1, e_2, e_3 of a face. Now consider the handle decomposition of M dual to the ideal triangulation. Then there is a one-to-one correspondence between 3-colorings and maps

$$\overline{c} : \{2-\text{handles}\} \to \mathbb{Z}_2,$$

and $c \in A_3$ if and only if $\overline{c}$ is a 2-cycle; that is if and only if $\overline{c} \in Z_2(M, \mathbb{Z}_2)$. Hence we get $|A_3| = \dim(Z_2(M, \mathbb{Z}_2))$. Since there are no 3-handles, $H_2(M, \mathbb{Z}_2) \cong Z_2(M, \mathbb{Z}_2)$. Therefore,

$$TV_3(M) = |A_3| = \dim(H_2(M, \mathbb{Z}_2)) = 2^{b_2(M)}.$$

$\square$

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