GROMOV NORM AND TURAEV-VIRO INVARIANTS OF 3-MANIFOLDS

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Abstract. We establish a relation between the "large r" asymptotics of the Turaev-Viro invariants TV_r and the Gromov norm of 3-manifolds. We show that for any orientable, compact 3-manifold M, with (possibly empty) toroidal boundary, $\log |TV_r(M)|$ is bounded above by a function linear in r and whose slope is a positive universal constant times the Gromov norm of M. The proof combines TQFT techniques, geometric decomposition theory of 3-manifolds and analytical estimates of $6j$-symbols.

We obtain topological criteria that can be used to check whether the growth is actually exponential; that is one has $\log |TV_r(M)| \geqslant B\, r$, for some $B > 0$. We use these criteria to construct infinite families of hyperbolic 3-manifolds whose $SO(3)$ Turaev-Viro invariants grow exponentially. These constructions are essential for the results of [9] where the authors make progress on a conjecture of Andersen, Masbaum and Ueno about the geometric properties of surface mapping class groups detected by the quantum representations.

We also study the behavior of the Turaev-Viro invariants under cutting and gluing of 3-manifolds along tori. In particular, we show that, like the Gromov norm, the values of the invariants do not increase under Dehn filling and we give applications of this result on the question of the extent to which relations between the invariants TV_r and hyperbolic volume are preserved under Dehn filling.

Finally we give constructions of 3-manifolds, both with zero and non-zero Gromov norm, for which the Turaev-Viro invariants determine the Gromov norm.

1. Introduction

Since the discovery of the quantum 3-manifold invariants in the late 80's, it has been a major challenge to understand their relations to the topology and geometry of 3-manifolds. Open conjectures predict tight connections between quantum invariants and the geometries coming from Thurston's geometrization picture [5, 6]. However, despite compelling physics and experimental evidence, progress to these conjectures has been scarce. For instance, the volume conjecture for the colored Jones polynomial has only been verified for a handful of hyperbolic knots to date. The reader is referred to [5] for survey articles on the subject and for related conjectures. On the other hand, coarse relations between the stable coefficients of colored Jones polynomials and volume have been established for an abundance of hyperbolic knots [8, 13, 15].

In this paper we are concerned with the question of how the "large level" asymptotics of the Turaev-Viro 3-manifold invariants relate to, and interact with, the geometric decomposition of 3-manifolds. The Turaev-Viro invariants $TV_r(M)$ of a compact oriented 3-manifold

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M are combinatorially defined invariants that can be computed from triangulations of M [32]. They are real valued invariants, indexed by a positive integer r, called the level, and for each r they depend on an $2r$-th root of unity. We combine TQFT techniques, geometric decomposition theory of 3-manifolds and analytical estimates of $6j$-symbols to show that, the r-growth of $TV_r(M)$ is bounded above by a function exponential in r that involves the Gromov norm of M.

We also obtain topological criteria for the growth to be exponential; that is to have $TV_r(M) \geqslant \exp Br$ with B a positive constant. We use these criteria to construct infinite families of hyperbolic 3-manifolds whose $SO(3)$-Turaev-Viro invariants grow exponentially. These results are used by the authors [9] to make progress on a conjecture of Andersen, Masbaum and Ueno (AMU Conjecture) about the geometric properties of surface mapping class groups detected by the quantum representations.

1.1. **Upper bounds.** For a compact oriented 3-manifold M, let $TV_r(M, q)$ denote the r-th Turaev-Viro invariant of M at root q, where q is a $2r$-th root of unity such that q^2 is a primitive r-th root of unity. Throughout the paper we will work with $q^2 = e^{\frac{2\pi i}{r}}$ and r an odd integer and we will often write $TV_r(M) := TV_r(M, e^{\frac{2\pi i}{r}})$. This is the theory that corresponds to the $SO(3)$ gauge group. We define,

$$LTV(M) = \limsup_{r \to \infty} \frac{2\pi}{r} \log |TV_r(M))| \tag{1}$$

where r runs over all odd integers. Also we will use $||M||$ to denote the Gromov norm (or simplicial volume) of M. See Section 2.1 of definitions. The main result of this article is the following.

Theorem 1.1. *There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold M with empty or toroidal boundary we have*

$$LTV(M) \leqslant C||M||.$$

If the interior of M admits a complete hyperbolic structure then, by Moscow rigidity, the hyperbolic metric is essentially unique and the volume of the metric is a topological invariant denoted by $\mathrm{vol}(M)$, that is essentially the Gromov norm. In this case, Theorem 1.1 provides a relation between hyperbolic geometry and the Turaev-Viro invariants. If M is the complement of a hyperbolic link in S^3 then we know that $lTV(M) \geqslant 0$ and in many instances the inequality is strict (Corollary 1.3).

The problem of estimating the volume of hyperbolic 3-manifolds in terms of topological quantities and quantum invariants, has been studied considerably in the literature. See for example [1, 13, 14] and references in the last item. Despite progress, to the best of our knowledge, Theorem 1.1 gives the first such linear lower bound that works for all hyperbolic 3-manifolds.

In the generality that Theorem 1.1 is stated, the constant C is about 8.3581×10^9. However, within classes of 3-manifolds, one has much more effective estimates. For instance, Theorem 7.4 of this paper shows that for most (in a certain sense) hyperbolic links $L \subset S^3$ we have

$$LTV(S^3 \smallsetminus L) \leqslant 10.5 \ \mathrm{vol}(S^3 \smallsetminus L).$$

Furthermore, given any constant E arbitrarily close to 1, one has infinite families of hyperbolic closed and cusped 3-manifolds M, with $LTV(S^3 \smallsetminus L) \leqslant E \operatorname{vol}(S^3 \smallsetminus L)$. See Section 7.2. for precise statements and more details.

We also give families of 3-manifolds with $LTV(M) = ||M||$. One such family of examples is the class of links with zero Gromov norm in S^3 or in $S^1 \times S^2$, but we also present families with non-zero norm (Section 8).

1.2. **Outline of proof of Theorem 1.1.** A major step in the proof is to show that $LTV(M)$ is finite for any compact oriented 3-manifold M. This is done by studying the large r asymptotic behavior of the quantum $6j$-symbols, and using the state sum formulae for the invariants TV_r. More specifically, we prove the following.

Theorem 1.2. *Suppose that M is a compact, oriented manifold with a triangulation consisting of t tetrahedra. Then, we have*

$$LTV(M) \leqslant 2.08 \; v_8 \; t,$$

where $v_8 \simeq 3.6638..$ is the volume of a regular ideal octahedron.

A second key argument we need is Theorem 5.2 of the paper which describes the behavior of the Turaev-Viro invariants under the operation of gluing or cutting 3-manifolds along tori. The proof of the theorem uses a version of a result of Roberts and Benedetti-Petronio that relates $TV_r(M, e^{\frac{2\pi i}{r}})$ to the $SO(3)$-Witten-Reshetikhin-Turaev invariants, and it relies heavily on the properties of the corresponding TQFT as constructed by Blanchet, Habegger, Masbaum and Vogel [4].

A third important ingredient in the proof of Theorem 1.1 is a theorem of Thurston asserting that given a hyperbolic 3-manifold M, after drilling out finitely many geodesics we obtain a 3-manifold admitting a triangulation with number of tetrahedra bounded above by a constant times $\operatorname{vol}(M)$. By the Geometrization Theorem, a compact oriented 3-manifold M, with possibly empty toroidal boundary can be cut along a canonical collection of tori into pieces that are either Seifert fibered manifolds or hyperbolic (JSJ-decomposition). We prove Theorem 1.1 by exploiting, by means of Theorem 5.2, compatibility properties of $||M||$ and LTV with the JSJ decomposition, and by studying separately LTV for Seifert fibered manifolds and hyperbolic manifolds.

1.3. **Lower bounds and the AMU Conjecture.** A very interesting problem, that we will not address in this paper, is to prove the opposite inequality of that given in Theorem 1.1. We will discuss the weaker problem of exponential r-growth of the invariants $TV_r(M)$. Define

$$lTV(M) = \liminf_{r \to \infty} \frac{2\pi}{r} \log |TV_r(M)|,$$

where r runs over all odd integers.

In Section 5 we show that, much like the Gromov norm, the value of the invariant $lTV(M)$ does not increase under the operation of Dehn filling and we discuss applications to the question of the extent to which relations between Turaev-Viro invariants and hyperbolic volume are preserved under Dehn filling. This in turn, leads a topological criterion for checking whether the invariants $TV_r(M)$, of a 3-manifold M, grow exponentially with respect to r. That is checking whether $lTV(M) > 0$. As a concrete application of this criterion, combined with a result of [10], we mention the following.

Corollary 1.3. *Let $M \subset S^3$ denote the complement of the figure-8 knot or the Borromean rings. For any link $L \subset M$ we have*

$$lTV(M \smallsetminus L) \geqslant 2v_3,$$

where $v_3 \sim 1.0149$ is volume of a regular ideal tetrahedron.

Since, by [10], the invariants $TV_r(S^3 \smallsetminus L)$ of a link complement are expressed in terms of the colored Jones polynomial of L, Corollary 1.3 also provides new instances of colored Jones polynomial values with exponential growth. As far as we know this is the first instance where exponential growth of a quantum type invariant follows from a topological argument rather than brute force computations. The result is consistent with the TV_r volume conjecture of Chen and Yang [6], which claims that for any hyperbolic 3-manifold of finite volume we should have $lTV(M) = LTV(M) = \mathrm{vol}(M)$. See Section 8 for more details.

Establishing exponential growth of the invariants TV_r is also important for another intriguing and wide-open conjecture in quantum topology, namely the AMU Conjecture [2]. In particular, Corollary 1.3 has an essential application to this conjecture that we will explain next.

For a compact orientable 3-manifold of genus g and n boundary components, say $\Sigma_{g,n}$, let $\mathrm{Mod}(\Sigma_{g,n})$ denote its mapping class group. The AMU conjecture asserts that the $SU(2)$ and $SO(3)$ quantum representations $\mathrm{Mod}(\Sigma_{g,n})$ should send pseudo-Anosov mapping classes to elements of infinite order (for large enough level). Despite good progress on the AMU Conjecture for low genus surfaces [2, 11, 27, 26], the first examples that satisfy the conjecture in surfaces of genus at least 2 were recently given by Marché and Santharoubane [21].

In [9] the authors show that if we have $lTV(M) > 0$ for all hyperbolic 3-manifolds that fiber over the circle, then the AMU Conjecture is true. Corollary 1.3 is then one of the key ingredients used in [9] to prove the following.

Theorem 1.4. ([9]) *Suppose that either $n = 2$ and $g \geqslant 3$ or $g \geqslant n \geqslant 3$. Then there are infinitely many non-conjugate pseudo-Anosov mapping classes in $\mathrm{Mod}(\Sigma_{g,n})$ that satisfy the AMU conjecture.*

As far as we know Theorem 1.4 is the first result that provides infinitely many mapping classes that satisfy the AMU conjecture for fixed surfaces of genus at least 2.

The paper is organized as follows: In Section 2, we recall some results about the simplicial volume and the geometric decomposition of 3-manifolds. In Section 3, we define the Turaev-Viro invariants and explain their TQFT properties. In Section 4.2 we provide a bound for quantum $6j$-symbols that are used to define to TV_r invariants, in terms of values of the Lobachevsky function. In Section 5 we study the behavior of LTV under the operations of cutting or gluing along tori. In Section 6, we study the special case of Seifert manifolds. In Section 7 we finish the proof of Theorem 1.1 and we derive Corollaries 7.2 and 1.3 and some generalizations. Finally, in Section 8, we provide some new examples where the growth rate of Turaev-Viro invariants exactly computes the simplicial volume.

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2. Decompositions of 3-manifolds

2.1. **Gromov norm preliminaries.** In this section, we recall the definition of the simplicial volume of a 3-manifold and some of its classical properties. Gromov defined simplicial volume of n-manifolds in [17], here we restrict ourselves to orientable 3-manifolds only. For more details the reader is referred to [29, Section 6.5].

Definition 2.1. [17, 29] Let M be a compact orientable 3-manifold with empty or toroidal boundary. Consider the fundamental class $[M, \partial M]$ in the singular homology $H_3(M, \partial M, \mathbb{R})$. For $z = \sum c_i\sigma_i \in Z_3(M, \partial M, \mathbb{R})$, a 3-relative singular cycle, representing $[M, \partial M]$, we define its norm to be the real number $||z|| = \sum |c_i|$.

(1) If $\partial M = \emptyset$, then the simplicial volume of M is

$$||M|| = \inf\{||z|| \ / \ [z] = [M]\}.$$

(2) If $\partial M \neq \emptyset$, the representative $[z] = [M, \partial M]$ determines a representative ∂z of $[\partial M] \in H_2(\partial M, \mathbb{R})$. Then, as shown in [29, Section 6.5] the following limit exists,

$$||M|| = \lim_{\varepsilon\to 0}\inf\{||z|| \ / \ [z] = [M, \partial M] \text{ and } ||\partial z|| \leqslant \varepsilon\},$$

and is defined to be the simplicial volume of M.

For hyperbolic manifolds, the simplicial volume is proportional to the hyperbolic volume and it is nicely behaved with respect to some topological operations.

Theorem 2.2. ([17, 29]) *The following are true:*

(1) $||M||$ *is additive under disjoint union and connected sums of manifolds.*
(2) *If M has a self map of degree $d > 1$ then $||M|| = 0$. In particular $||\Sigma \times S^1|| = 0$, for any compact oriented surface Σ.*
(3) *If T is an embedded torus in M and M' is obtained from M by cutting along T then*

$$||M|| \leqslant ||M'||,$$

Moreover, the inequality is an equality if T is incompressible in M.

(4) *If M is obtained from M' by Dehn-filling of a torus boundary component in M', then*

$$||M|| \leqslant ||M'||,$$

(5) *If M has a complete hyperbolic structure with finite volume then*

$$\mathrm{vol}(M) = v_3||M||.$$

2.2. **Geometric decomposition.** We recall that any compact oriented 3-manifold is a connected sum of irreducible manifolds and copies of $S^2 \times S^1$. Furthermore, by the Jaco-Shalen-Johannson (JSJ) theorem, any irreducible 3-manifold M can be cut along a collection of incompressible tori $\mathcal{T} = \{T_1, \ldots, T_n\}$, so that the components of $M \smallsetminus (T_1 \cup \ldots \cup T_n)$ are irreducible atoroidal 3-manifolds.

Thurston's Geometrization Conjecture [30], proved by Perelman, allows one to identify the pieces of the JSJ decomposition. For details the reader is referred to books by Tian and Morgan A consequence of Perelman's work is the following theorem which is the solution to Thurston's Geometrization Conjecture [23].

Theorem 2.3. (Geometrization Theorem, [23]) *Any irreducible compact orientable* 3*-manifold M contains a unique (up to isotopy) collection of disjointly embedded incompressible tori $\mathcal{T} = \{T_1, \ldots, T_n\}$ such that all the connected components of $M \smallsetminus \mathcal{T}$ are either Seifert fibered manifolds or hyperbolic.*

2.3. **Efficient bounds on triangulations of 3-manifolds.** We conclude this section by recalling a result about triangulations of 3-manifolds. As the Turaev-Viro invariants of a manifold M are defined using state sums whose terms are products of quantum $6j$-symbols over a triangulation of M, we wish to use triangulations with few tetrahedra to bound the Turaev-Viro invariants. For hyperbolic 3-manifolds, one way to achieve this is to consider triangulations not of the manifold M itself, but rather of M minus some geodesics. We will use the following theorem, due to W. Thurston, originally used in the proof of the so called Jorgensen-Thurston Theorem [29, Theorem 5.11.2].

Theorem 2.4. (Thurston) *There exists a universal constant C_2, such that for any complete hyperbolic* 3*-manifold M of finite volume, there exists a link L in M and a partially ideal triangulation of $M \smallsetminus L$ with less than $C_2||M||$ tetrahedra.*

The proof of Theorem 2.4 comes from the thick-thin decomposition of hyperbolic manifolds. Moreover, the constant C_2 in this theorem can be explicitly estimated:

It follows from the analysis in the proof of [29, Theorem 5.11.2] that given $\varepsilon \leqslant \frac{c}{2}$, where c is the Margulis constant, one can choose

$$C_2 = \frac{\binom{k}{3} v_3}{4V(\varepsilon/4)} \quad \text{where} \quad k = \lfloor \frac{V(5\varepsilon/4)}{V(\varepsilon/4)} \rfloor - 1,$$

where $V(r)$ denotes the hyperbolic volume of a ball of radius r. The volume $V(r)$ be can be computed by the formula $V(r) = \pi\left(\sinh(2r) - 2r\right)$ (see, for example, [18, Section 3.1]). Moreover, the Margulis constant has been shown to be at least at least 0.104 [22]. Using $\varepsilon = \frac{0.103}{2}$ we get that in Theorem 2.4, we can use $C_2 = 1.101 \times 10^9$.

3. Turaev-Viro invariants, Reshetikhin-Turaev invariants and TQFT

In this section we summarize the definitions and the main properties of the quantum invariants we will use in this paper. First we recall the definition of the Turaev-Viro invariants as state sums on triangulations of 3-manifolds. Then in subsection 3.2 we summarize the properties of the $SO(3)$-Reshetikhin-Turaev TQFT [4, 25, 31] that we will need in this paper.

3.1. **State sums for the Turaev-Viro invariants.** Let $r \geqslant 3$ be an odd integer and let $q = e^{\frac{2i\pi}{r}}$. Define the quantum integer $\{n\}$ by

$$\{n\} = q^n - q^{-n} = 2\sin(\frac{2n\pi}{r}) = 2\sin(\frac{2\pi}{r})[n], \quad \text{where} \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{2\sin(\frac{2n\pi}{r})}{2\sin(\frac{2\pi}{r})},$$

and define the quantum factorial by $\{n\}! = \prod_{i=1}^{n} \{i\}$.

Consider the set $I_r = \{0, 2, 4 \ldots, r-3\}$ of all non-negative even integers less than $r-2$. A triple (a_i, a_j, a_k) of elements in I_r, is called *a*dmissible if $a_i + a_j + a_k \leqslant 2(r-2)$ and we have triangle inequalities $a_i \leqslant a_j + a_k$, $a_j \leqslant a_i + a_k$, and $a_k \leqslant a_i + a_j$. For an admissible triple we define $\Delta(a_i, a_j, a_k)$ by

$$\Delta(a_i, a_j, a_k) = \zeta_r^{\frac{1}{2}} \left(\frac{\{\frac{a_i+a_j-a_k}{2}\}!\{\frac{a_j+a_k-a_i}{2}\}!\{\frac{a_i+a_k-a_j}{2}\}!}{\{\frac{a_i+a_j+a_k}{2}+1\}!} \right)^{\frac{1}{2}}$$

where $\zeta_r = 2\sin(\dfrac{2\pi}{r})$. A 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6) \in I_r^6$ is called admissible if each of the triples

$$F_1 = (a_1, a_2, a_3), \quad F_2 = (a_2, a_4, a_6), \quad F_3 = (a_1, a_5, a_6) \quad \text{and} \quad F_4 = (a_3, a_4, a_5), \tag{2}$$

is admissible. Given an admissible 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$, we define the quantum $6j$-symbol at the root q by the formula

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{vmatrix} = (\zeta_r)^{-1}(\sqrt{-1})^{\lambda} \prod_{i=1}^{4} \Delta(F_i) \sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}} \frac{(-1)^z\{z+1\}!}{\prod_{j=1}^4 \{z-T_j\}! \prod_{k=1}^3 \{Q_k - z\}!} \tag{3}$$

where $\lambda = \sum_{i=1}^{6} a_i$, and

$$T_1 = \frac{a_1+a_2+a_3}{2}, \quad T_2 = \frac{a_1+a_5+a_6}{2}, \quad T_3 = \frac{a_2+a_4+a_6}{2} \quad \text{and} \quad T_4 = \frac{a_3+a_4+a_5}{2},$$

$$Q_1 = \frac{a_1+a_2+a_4+a_5}{2}, \quad Q_2 = \frac{a_1+a_3+a_4+a_6}{2} \quad \text{and} \quad Q_3 = \frac{a_2+a_3+a_5+a_6}{2}.$$

Definition 3.1. An admissible coloring of a tetrahedron Δ is an assignment of an admissible 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ of elements of I_r to the edges of Δ so that the three numbers assigned to the edges of each face form an admissible triple. In this setting, the quantities T_i and Q_i defined above correspond to the sums of colorings over faces of the tetrahedron, and the sums of colorings of edges of normal quadrilaterals in Δ.

Given a compact orientable 3-manifold M consider a triangulation τ of M. If $\partial M \neq \emptyset$ we will allow τ to be a (partially) ideal triangulation, where some of the vertices of the tetrahedra are truncated and the truncated faces triangulate ∂M. Given a partially ideal triangulation τ the set V of interior vertices of τ is the set of vertices of τ which do not lie on ∂M. Also we write E for the set of interior edges (thus excluding edges coming from the truncation of vertices). A *coloring at level* r of the triangulated 3-manifold (M, τ) is an assignment of elements of I_r to the edges of τ and is *admissible* if the 6-tuple assigned to the edges of each tetrahedron of τ is admissible. Let c be an admissible coloring of (M, τ) at level r. Given a coloring c and an edge $e \in E$ let $|e|_c = (-1)^{c(e)}[c(e)+1]$. Also for Δ a tetrahedron in τ let $|\Delta|_c$ be the quantum $6j$-symbol corresponding to the admissible 6-tuple assigned to Δ by c. Finally, $A_r(\tau)$ denote the set of r-admissible colorings of τ and let $\eta_r = \dfrac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}$. We are now ready to define the Turaev-Viro invariants as a state-sum over $A_r(\tau)$:

Theorem 3.2. ([19, 32]) *Let M be a compact orientable manifold closed or with boundary. Let b_2 denote the second $\mathbb{Z}_2$-Betti number of M and b_0 is the number of closed connected components in M. Then the state sum*

$$TV_r(M) = 2^{b_2-b_0}\ \eta_r^{2|V|} \sum_{c\in A_r(\tau)} \prod_{e\in E} |e|_c \prod_{\Delta\in\tau} |\Delta|_c, \tag{4}$$

is independent of the partially ideal triangulation τ of M, and thus defines a topological invariant of M.

Note that while the definition above differs slightly from [19, Definition 7] by the use of even colors only, by [10, Theorem 2.9] the two definitions are essentially the same; they only differ by the factor of $2^{b_2-b_0}$. The restriction of the coloring set to only even integers is reminiscent to the $SO(3)$ quantum invariant theory and it facilitates the study of the Turaev-Viro invariants, for odd levels, via the $SO(3)$-TQFT theory of [4].

3.2. **Reshetikhin-Turaev invariants and TQFT.** The definition of the Turaev-Viro invariants given above will be useful for us to show that the upper limit $LTV(M)$ is well defined (i.e. it is finite). However, in order to understand the topological properties of $LTV(M)$ (i.e. its behavior under prime and toroidal decompositions of 3-manifolds) it will be convenient for us to view Turaev-Viro invariants through their relation to the Reshetikhin-Turaev invariants $RT_r(M)$ ([24]), and the Reshetikhin-Turaev Topological Quantum Field Theories (TQFTs) they are part of.
The Reshetikhin-Turaev TQFTs are functors from the category of cobordisms in dimension $2+1$ to the category of finite dimensional vector spaces; they associate a finite dimensional $\mathbb{C}$-vector space $RT_r(\Sigma)$ to each compact oriented surface Σ, while their values on closed 3-manifolds M are the Reshetikhin-Turaev invariants $RT_r(M)$ which are $\mathbb{C}$-valued and are related to surgery presentations of 3-manifolds and colored Jones polynomials.
Also, for M with boundary $\partial M = \Sigma$, $RT_r(M) \in RT_r(\Sigma)$ is a vector.
We will introduce these TQFTs in the skein-theoretic framework of Blanchet, Habegger, Masbaum and Vogel [4]. As we restrict to level r odd, the TQFTs we are using are the so called $SO(3)$-TQFTs. Below we will sketch only the properties of these TQFTs we need, referring to [4] for a precise definition.

To fix some notations, we recall that when V is a $\mathbb{C}$-vector space, $\overline{V}$ denotes the $\mathbb{C}$-vector space that is V as an abelian group and whose scalar multiplication is $\alpha \cdot v = \overline{\alpha} v$. When V is a $\mathbb{C}$-vector space, an Hermitian form $\langle \cdot,\cdot\rangle$ on V is a map

$$\langle \cdot,\cdot\rangle : V \otimes V \to \mathbb{C},$$

that satisfies $\langle \alpha v + w, v'\rangle = \alpha\langle v, v'\rangle + \langle w, v'\rangle$ and $\langle w, v\rangle = \overline{\langle v, w\rangle}$. Note that an Hermitian form can be considered a bilinear form over $V \otimes \overline{V}$.

Remark 3.3. We note that the invariants $RT_r(M)$ in [4] are only well-defined up to a $2r$-th root of unity, this ambiguity being called the anomaly of the TQFT. Resolving the anomaly requires considering 3-manifolds M with an additional structure called a p_1-structure, see

[4] for details. As we will only be interested in the moduli of the $RT_r(M)$, we will neglect the anomaly. We warn the reader, however, that the rules for computing RT_r in Theorem 3.4 below have to be understood to hold up to a root of unity.

We summarize the main properties of the SO(3)-TQFT defined in [4] in the following theorem:

Theorem 3.4. ([4, Theorem 1.4]) *Let r be an odd integer and A be a primitive $2r$-th root of unity. Then there exists a TQFT functor RT_r in dimension 2+1 satisfying:*

(1) *For any oriented compact closed 3-manifold M, $RT_r(M) \in \mathbb{C}$ is a topological invariant. Moreover if $\overline{M}$ is the manifold M with the opposite orientation, then $RT_r(\overline{M}) = \overline{RT_r(M)}$.*
(2) *We have $RT_r(S^2 \times S^1) = 1$ and $RT_r(S^3) = \eta_r = \dfrac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}$.*
(3) *The invariants RT_r are multiplicative under disjoint union of 3-manifolds, and for connected sums we have*
$$RT_r(M \# M') = \eta_r^{-1} RT_r(M) RT_r(M').$$
(4) *For any closed compact oriented surface Σ, $RT_r(\Sigma) = V_r(\Sigma)$ is a finite dimensional $\mathbb{C}$-vector space and for disjoint unions we have natural isomorphisms*
$$V_r(\Sigma_1 \coprod \Sigma_2) \simeq V_r(\Sigma_1) \otimes V_r(\Sigma_2),$$
Moreover, $V_r(\emptyset) = \mathbb{C}$ and for any oriented surface $V_r(\overline{\Sigma})$ is the $\mathbb{C}$-vector space $\overline{V_r(\Sigma)}$.
(5) *To every compact oriented 3-manifold M with a fixed homeomorphism $\partial M \simeq \Sigma$ there is an associated vector $RT_r(M) \in V_r(\Sigma)$. Moreover for a disjoint union $M = M_1 \coprod M_2$, we have*
$$RT_r(M) = RT_r(M_1) \otimes RT_r(M_2) \in V_r(\Sigma_1) \otimes V_r(\Sigma_2).$$
(6) *For any odd integer r, there is a natural Hermitian form*
$$\langle \cdot, \cdot \rangle : V_r(\Sigma) \otimes V_r(\overline{\Sigma}) \to \mathbb{C},$$
with the following property: Given M a compact oriented 3-manifold and Σ an embedded surface in M, if we let M' be the manifold obtained by cutting M along Σ, with $\partial M' = \Sigma \coprod \overline{\Sigma} \coprod \partial M$, then we have $RT_r(M) = \Phi(RT_r(M'))$. Here Φ is the linear map
$$\Phi : V_r(\Sigma) \otimes V_r(\overline{\Sigma}) \otimes V_r(\partial M) \longrightarrow V_r(\partial M),$$
defined by $\Phi(v \otimes w \otimes \varphi) = \langle v, w \rangle \varphi$.

Mapping Cylinders. A class of 3-manifolds with boundary to which the construction can be applied are the mapping cylinders of maps of surfaces: Given a surface Σ and an element $\varphi \in \mathrm{Mod}(\Sigma)$ in its mapping class group, let
$$M_\varphi = [0,1] \times \Sigma \underset{(x,1) \sim \varphi(x)}{\cup} \Sigma.$$

Then $RT_r(M_\varphi)$ is a vector in $V_r(\Sigma) \otimes \overline{V_r(\Sigma)}$. The later space can be identified with $\mathrm{End}(V_r(\Sigma))$ as $V_r(\overline{\Sigma}) \simeq V_r(\Sigma)^*$ by the natural Hermitian form. The assignment $\rho_r(\varphi) = RT_r(M_\varphi)$, defines a projective representation

$$\rho_r : \mathrm{Mod}(\Sigma) \longrightarrow \mathbb{P}\mathrm{End}(V_r(\Sigma)).$$

These representations are known as the the quantum representations of mapping class groups; they are projective because of the above mentioned TQFT anomaly factor. Given the mapping torus

$$N_\varphi = [0,1] \times \Sigma/_{(x,1)\sim(\varphi(x),0)},$$

of a class $\varphi \in \mathrm{Mod}(\Sigma)$, by [4, Formula 1.2] we have $RT_r(N_\varphi) = \mathrm{Tr}(\rho_r(\varphi))$. We will need the following well-known fact which we state as a lemma.

Lemma 3.5. *Let T^2 be the 2-dimensional torus and let $\varphi : T^2 \simeq S^1 \times S^1 \to S^1 \times S^1$, be the elliptic involution, defined by $(x, y) \to (-x, -y)$. Then $\rho_r(\varphi) = \mathrm{id}_{V_r(T^2)}$.*

In the next statement we summarize from [4] the facts about the dimensions of $V_r(\Sigma)$ that we will need.

Theorem 3.6. ([4]) *We have the following:*

1. *For any odd integer $r \geqslant 3$, and any primitive $2r$-th root of unity, the $V_r(T^2)$ has dimension $\frac{r-1}{2}$ and the Hermitian form $\langle \cdot, \cdot \rangle$ on $V_r(T^2)$ is definite positive.*
2. *If Σ_g is the closed compact oriented surface of genus $g \geqslant 2$, then $\dim(V_r(\Sigma_g))$ is a polynomial in r of degree $3g - 3$.*

Proof. The first assertion of part (1) is proved in [4, Corollary 4.10] and the second assertion is given in [4, Remark 4.12]. The second assertion follows by [4, Corollary 1.16 and Remark(iii)]. □

To continue recall that the double $D(M)$ of a manifold M is defined as $M \coprod \overline{M}$ if M is closed and as $M \underset{\Sigma}{\cup} \overline{M}$ if M has non-empty boundary. We end this section with a theorem that for a manifold M relates the $SO(3)$-Turaev-Viro invariants $TV_r(M)$ defined in Section 3.1, to the RT_r invariant of the double $D(M)$ of M.

Theorem 3.7. ([3]) *Let M be a 3-manifold with boundary, r be an odd integer and $q = e^{\frac{2i\pi}{r}}$. Then*

$$TV_r(M, q) = \eta_r^{-\chi(M)} RT_r(D(M), e^{\frac{i\pi}{r}}),$$

where $\chi(M)$ is the Euler characteristic of M and $\eta_r = \dfrac{2\sin(\frac{2\pi}{r})}{\sqrt{r}} = RT_r(S^3)$.

The proof of Theorem 3.7 for closed manifolds is due to Roberts [25] and Walker and Turaev [31]. The proof for manifolds with non-empty boundary is essentially due to Benedetti and Petronio [3]: Although in [3] only the invariants $RT_r(M)$ corresponding to the SU(2)-TQFT are considered, the proof can be adapted is the $SO(3)$-TQFT setting. This was done in [10, Theorem 3.1].

4. Finiteness of LTV

The goal of this section is to prove Theorem 1.2. We first provide an upper bound to quantum $6j$-symbols at level r and at the root $q = e^{\frac{2i\pi}{r}}$. Using this we bound the invariants $TV_r(M)$ of a 3-manifold M in terms of the number of tetrahedra in a triangulation of M. This, in particular, will prove that $LTV(M)$ is finite.

4.1. **Quantum factorials and the Lobachevsky function.** It is a well known fact in quantum topology that the asymptotics of quantum factorials are related to the Lobachevsky function. In this section we give a version of this fact at the root $q = e^{\frac{2i\pi}{r}}$.
For P a Laurent polynomial, let $ev_r(P)$ be the evaluation of the absolute value of P at $q = e^{\frac{2i\pi}{r}}$, that is

$$ev_r(P) = |P(e^{\frac{2i\pi}{r}})|.$$

Let also $\Lambda(x)$ denote the Lobachevski function, defined by

$$\Lambda(x) = -\int_0^x \log|2\sin(x)|dx.$$

We estimate the growth of quantum factorials at $q = e^{\frac{2i\pi}{r}}$ using the following lemma.

Proposition 4.1. *Given an integer $0 < n < r$ we have*

$$\log(ev_r(\{n\}!)) = -\frac{r}{2\pi}\Lambda(\frac{2n\pi}{r}) + O(\log r).$$

Moreover, in this estimate the $O(\log r)$ is uniform: there exists a constant C_3 independent of n and r, such that $O(\log r) \leqslant C_3 \log r$.

Proof. First note that $ev_r(\{n\}!) = \prod\limits_{j=1}^{n} 2\sin(\frac{2j\pi}{r})$. In this product as r is odd and $0 < n < r$, all factors are non-zero. Thus we can write

$$\log(ev_r(\{n\}!)) = \sum_{j=1}^{n} \log|2\sin(\frac{2j\pi}{r})|.$$

The function

$$f(t) = \log|2\sin(\frac{2\pi t}{r})|,$$

is differentiable on $(0, \frac{r}{2})$ and $(\frac{r}{2}, r)$.
Case 1: Assume that $n < \frac{r}{2}$. The Euler-Mac Laurin formula gives that

$$\log(ev_r(\{n\}!)) = \int_1^n \log|2\sin(\frac{2\pi t}{r})| + \frac{f(1)+f(n)}{2} + R_0,$$

where the $R_0 \leqslant \frac{1}{2}\int_1^n |f'(t)|dt \leqslant 2 \sup\limits_{t\in[1,n]} |f(t)|$ as f is increasing then decreasing on $(0, \frac{r}{2})$.

Note that $|n - \frac{r}{2}| > \frac{1}{2}$ and $\sin(t) \geqslant \frac{2t}{\pi}$ for $0 \leqslant t \leqslant \frac{\pi}{2}$. Hence, the quantities $|f(1)|, |f(n)|$ and $\sup\limits_{t\in[1,n]} |f(t)|$ are all bounded by $|\log(\frac{4}{r})|$ for big r. Moreover, for big r, we have

$$\left|\int_0^1 \log|2\sin(\frac{2\pi t}{r})|dt\right| \leqslant \left|\int_0^1 \log(\frac{4t}{r})dt\right| \leqslant |\log r| + \left|\int_0^1 \log(4t)dt\right|.$$

Thus

$$\log(ev_r(\{n\}!)) = \int_0^n \log|2\sin(\frac{2\pi t}{r})| + O(\log r)$$
$$= \frac{r}{k\pi}\int_0^{\frac{2n\pi}{r}} \log|2\sin t| + O(\log r) = -\frac{r}{2\pi}\Lambda(\frac{2n\pi}{r}) + O(\log r).$$

Note that in all our estimations the $O(\log r)$ was independent on $0<n<\frac{r}{2}$.

Case 2: Assume that $n \geqslant \frac{r}{2}$. Now we write $\log(ev_r(\{n\}!))$ as a sum of two terms

$$\log(ev_r(\{n\}!)) = \sum_{j=1}^{\lfloor \frac{r}{2}\rfloor} \log|2\sin(\frac{2j\pi}{r})| + \sum_{j=\lceil \frac{r}{2}\rceil}^{n} \log|2\sin(\frac{2j\pi}{r})|,$$

Applying the Euler-Mac Laurin formula for each sum we get

$$\log(ev_r(\{n\}!)) = \int_1^{\lfloor \frac{r}{2}\rfloor} \log|2\sin(\frac{2\pi t}{r})|dt + \int_{\lceil \frac{r}{2}\rceil}^{n} \log|2\sin(\frac{2\pi t}{r})|dt$$
$$+ \frac{f(1)+f(\lfloor \frac{r}{2}\rfloor)+f(\lceil \frac{r}{2}\rceil)+f(n)}{2} + R_0,$$

where

$$R_0 \leqslant \frac{1}{2}\left(\int_1^{\lfloor \frac{r}{2}\rfloor} |f'(t)|dt + \int_{\lceil \frac{r}{2}\rceil}^{n} |f'(t)|dt\right) \leqslant 2\left(\sup_{t\in[1,\lfloor \frac{r}{2}\rfloor]}|f(t)| + \sup_{t\in[\lceil \frac{r}{2}\rceil,n]}|f(t)|\right),$$

as f is increasing then decreasing on $(0,\frac{r}{2})$ and increasing then decreasing on $(\frac{r}{2},r)$. Since r is odd and $\frac{r}{2}$ is a half integer, similarly as in Case 1, we have that $f(1), f(\lfloor \frac{r}{2}\rfloor), f(\lceil \frac{r}{2}\rceil), f(n)$ are all $\leqslant C_3\log r$ for some C_3 independent of n, and also

$$\int_1^{\lfloor \frac{r}{2}\rfloor} \log|2\sin(\frac{2\pi t}{r})|dt + \int_{\lceil \frac{r}{2}\rceil}^{n} \log|2\sin(\frac{2\pi t}{r})|dt$$
$$= \int_0^n \log|2\sin(\frac{2\pi t}{r})|dt + O(\log r) = -\frac{r}{2\pi}\Lambda(\frac{2n\pi}{r}) + O(\log r).$$

This concludes the proof of Proposition 4.1. □

4.2. **Upper bounds for quantum $6j$-symbols.** In this section, we find an upper bound for the quantum $6j$-symbols that we will need for the proof of Theorem 1.2. We show the following:

Proposition 4.2. *For any r-admissible 6-tuple (a,b,c,d,e,f), we have that*

$$\frac{2\pi}{r}\log\left(ev_r\begin{vmatrix} a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\end{vmatrix}\right) \leqslant v_8 + 8\Lambda(\frac{\pi}{8}) + O(\frac{\log r}{r}).$$

Moreover, in this estimate the $O(\frac{\log r}{r})$ is uniform: there exists a constant C_3 independent of the a_i's and r, such that $O(\frac{\log r}{r}) \leqslant C_3\frac{\log r}{r}$.

Remark 4.3. The above bound of $v_8+8\Lambda(\frac{\pi}{8})$ is not expected to be the optimal bound. In a closely related context, Costantino proved that the growth rates of quantum $6j$-symbols with so-called hyperbolic admissibility conditions are given by volumes of hyperbolic truncated tetrahedra [7]. The argument of Costantino is also applicable in our context from some (but not all) values of the a_i's. In the end, we expect the maximum growth rate to be the maximum volume of a hyperbolic truncated tetrahedron, which is v_8.

Proof. By Equation 3, we have

$$\log\left(ev_r\begin{vmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\end{vmatrix}\right)=\sum_{i=1}^{4}\log|\Delta(F_i)|+\log|S|+O(\log r),$$

where

$$S=\sum_{z=\max\{T_1,T_2,T_3,T_4\}}^{\min\{Q_1,Q_2,Q_3\}}\frac{(-1)^z\{z+1\}!}{\prod_{j=1}^4\{z-T_j\}!\prod_{k=1}^3\{Q_k-z\}!},$$

and F_1,F_2,F_3,F_4 are defined in equation (2). Each of the Δ terms is a product of 4 quantum factorials. Moreover as $Q_j-T_i\leqslant(r-2)$ for all i and j, we have

$$log|S|\leqslant\max_{\max T_i\leqslant z\leqslant\min Q_j}\log\left|\frac{\{z\}!}{\prod_{j=1}^4\{z-T_j\}!\prod_{k=1}^3\{Q_k-z\}!}\right|+O(\log r).$$

Here we used that the sum S is a sum of a polynomial number of terms and that for $0\leqslant z\leqslant r-2$, we have $\log|\{z+1\}|\leqslant O(\log r)$ for some $O(\log r)$ independent on z.
To estimate these terms we will use Proposition 4.1. We will write

$$A_i=\frac{2\pi a_i}{r},\ U_j=\frac{2\pi T_j}{r},\ \text{and } V_k=\frac{2\pi Q_k}{r}.$$

We have

$$\begin{aligned}(5)\quad & log\left(ev_r\begin{vmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\end{vmatrix}\right)\\ &\leqslant\frac{r}{2\pi}\left(v(\frac{A_1}{2},\frac{A_2}{2},\frac{A_3}{2})+v(\frac{A_1}{2},\frac{A_5}{2},\frac{A_6}{2})+v(\frac{A_2}{2},\frac{A_4}{2},\frac{A_6}{2})+v(\frac{A_3}{2},\frac{A_4}{2},\frac{A_5}{2})\right)\\ &\qquad+\frac{r}{2\pi}\max_Z g(Z,A_i)+O(\log r),\end{aligned}$$

where

$$v(\alpha,\beta,\gamma)=\frac{1}{2}\left(\Lambda\left(\alpha+\beta+\gamma\right)-\Lambda\left(\beta+\gamma-\alpha\right)-\Lambda\left(\alpha+\gamma-\beta\right)-\Lambda\left(\alpha+\beta-\gamma\right)\right),$$

and

$$g(Z,A_i)=\sum_{i=1}^{4}\Lambda(Z-U_i)+\sum_{j=1}^{3}\Lambda(V_i-Z)-\Lambda(Z).$$

Since the function Λ is bounded, the functions v and g also and thus

$$\log\left(ev_r\begin{vmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\end{vmatrix}\right)\leqslant\frac{r}{2\pi}C_1+O(\log r),$$

for some constant C_1. We show that one can use $C_1 = v_8 + 8\Lambda\left(\frac{\pi}{8}\right)$ by computing the maximum of functions v and g above. This is done using the following lemma that we will prove in the Appendix.

Lemma 4.4. *The maximum of the function v is $\dfrac{v_8}{4}$ and the maximum of the function g is $8\Lambda\left(\dfrac{\pi}{8}\right)$*

□

4.3. *LTV* **and state-sums.** Using state sums for the Turaev-Viro invariants and Proposition 4.2 we prove Theorem 1.2 stated in the Introduction.

Theorem 1.2. *Suppose that M is a compact, oriented manifold with a triangulation consisting of t tetrahedra. Then, we have*

$$LTV(M) \leqslant 2.08\ v_8\ t,$$

where $v_8 \simeq 3.6638..$ is the volume of a regular ideal octahedron.

Proof. Let τ be a triangulation of M with t tetrahedra. Recall that by equation (4) in the statement of Theorem 3.2

$$TV_r(M) = 2^{b_2-b_0}\ \eta_r^{2|V|} \sum_{c\in A_r(\tau)} \prod_{e\in E} |e|_c \prod_{\Delta\in\tau} |\Delta|_c.$$

Since the term $2^{b_2-b_0}$ is independent of r we may ignore it. Recall that $A_r(\tau)$ is the set of admissible r-colorings of the edges of the triangulation τ. The number of elements of the set $A_r(\tau)$ is bounded by a polynomial in r as each edge must be colored by an element of $\{0, 2, \dots r-3\}$. So

$$\frac{2\pi}{r}\log|TV_r(M)| \leqslant \frac{2\pi}{r}\log\left(|A_r(\tau)||P|\right) \leqslant \frac{2\pi}{r}\log|P| + O(\frac{\log r}{r}),$$

where P is the term in the sum of maximal $\log$. Moreover,

$$\log|\eta_r| = \log\left|\frac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}\right| = O(\log r),$$

and for the factors $|e|_c$ we have:

$$\log|e|_c = \log\left|\frac{\sin(\frac{2\pi(c(e)+1)}{r})}{\sin(\frac{2\pi}{r})}\right| = O(\log r).$$

Finally, by Proposition 4.2, for any $c \in A_r(\tau)$ and any $\Delta \in \tau$, the factor $\frac{2\pi}{r}\log|\Delta|_c$ is bounded above by $C_1 = v_8 + 8\Lambda\left(\frac{\pi}{8}\right) \simeq 7.5914 < 7.6207 \simeq 2.08 v_8$ and P has t such factors. The theorem follows. □

5. Cutting along tori

In this section, we will prove a theorem (Theorem 5.2 below) that describes the behavior of the Turaev-Viro invariants when cutting along a torus. This will follow from the TQFT properties of TV_r and Cauchy-Schwarz type inequalities. We will need the lemma.

Lemma 5.1. *Let $v_1, v_2, \ldots, v_n$ be vectors in the $\mathbb{C}$-vector space V with positive definite Hermitian form $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$. Then we have*

$$||\sum_{i=1}^{n} v_i||^2 \leqslant n \sum_{i=1}^{n} ||v_i||^2.$$

Proof. Let $e_1, \ldots, e_d$ be an orthonormal basis of $\mathrm{Span}(v_1, \ldots, v_n)$ and write

$$v_i = \sum_{j=1}^{d} \lambda_{ij} e_j.$$

Then

$$||\sum_{i=1}^{n} v_i||^2 = \sum_{j=1}^{d} |\sum_{i=1}^{n} \lambda_{ij}|^2 \leqslant \sum_{j=1}^{d} n \sum_{i=1}^{n} |\lambda_{ij}|^2 = n \sum_{i=1}^{n} ||v_i||^2,$$

where the inequality follows from the Cauchy-Schwarz inequality in $\mathbb{C}^n$. $\square$

Theorem 5.2. *Let $r \geqslant 3$ be an odd integer and let M be a compact oriented 3-manifold with empty or toroidal boundary. Let $T \subset M$ be an embedded torus and let M' be the manifold obtained by cutting M along T. Then*

$$TV_r(M) \leqslant \left(\frac{r-1}{2}\right) TV_r(M'),$$

and

$$LTV(M) \leqslant LTV(M').$$

If moreover T is separating then $TV_r(M) \leqslant TV_r(M')$.

Proof. Let $\Sigma = \partial M$. We will distinguish two cases:

Case 1: Suppose that the torus T is non-separating. The torus T inherits an orientation from that of M. The manifold M', obtained by cutting M along T, has boundary $\partial M' = \Sigma \coprod T \coprod \overline{T}$, where $\overline{T}$ is the torus T with opposite orientation. With the notation of Theorem 3.4, we have $RT_r(M') \in V_r(\partial M') = V_r(\Sigma) \otimes V_r(T) \otimes V_r(\overline{T})$ and $RT_r(M) \in V_r(\Sigma)$. Furthermore $RT_r(M) = \Phi(RT_r(M'))$, where Φ is the the contraction map of Theorem 3.4(6): We have $\Phi : V_r(\Sigma) \otimes V_r(T) \otimes V_r(\overline{T}) \to V_r(\Sigma)$, where

$$\Phi(v \otimes w_1 \otimes w_2) = \langle w_1, w_2 \rangle v.$$

By Theorem 3.7, we have

$$TV_r(M') = \langle RT_r(M'), RT_r(M') \rangle = ||RT_r(M')||^2 \ \text{ and } \ TV_r(M) = ||RT_r(M)||^2.$$

By hypothesis, Σ is a (possibly empty) union of, say n, tori; thus we have $V_r(\Sigma) = V_r(T^2)^{\otimes n}$ and $V_r(\partial M') = V_r(T^2)^{\otimes (n+2)}$. By Theorem 3.6 the Hermitian form on $V_r(T^2)$

is definite positive. Hence, setting $m = \frac{r-1}{2}$, we have an orthonormal basis $(\varphi_i)_{1\leqslant j\leqslant m}$ on $V_r(T^2)$. Using this basis we can write

$$RT_r(M') = \sum_{1\leqslant i,j\leqslant m} v_{ij}\otimes\varphi_i\otimes\varphi_j,$$

where the v_{ij} are vectors in $V_r(\Sigma)$ which is also an Hermitian vector space with definite positive Hermitian form. We have that

$$TV_r(M') = ||RT_r(M')||^2 = \sum_{1\leqslant i,j\leqslant m} ||v_{ij}||^2 \tag{6}$$

as $\varphi_i\otimes\varphi_j$ is an orthonormal basis of $V_r(T)\otimes V_r(\overline{T})$. On the other hand, applying the contraction map Φ, we get:

$$RT_r(M) = \Phi(RT_r(M')) = \sum_{1\leqslant i,j\leqslant m}\langle\varphi_i,\varphi_j\rangle v_{ij} = \sum_{1\leqslant i\leqslant m} v_{ii},$$

as φ_i is an orthonormal basis of $V_r(T)$. Thus

$$TV_r(M) = ||RT_r(M)||^2 = \left|\left|\sum_{1\leqslant i\leqslant m} v_{ii}\right|\right|^2 \leqslant m\sum_{1\leqslant i\leqslant m}||v_{ii}||^2 \leqslant m\sum_{1\leqslant i,j\leqslant m}||v_{ij}||^2 = mTV_r(M'),$$

where the first inequality follows from Lemma 5.1 and the last equality from Equation (6). This proves the first part of Theorem 5.2.

Case 2: Let T be a separating torus and let M_1 and M_2 be the two components of $M\smallsetminus T$. Let us write $\partial M_1 = T\cup\Sigma_1$ and $\partial M_2 = \overline{T}\cup\Sigma_2$, where Σ_1 and Σ_2 are actually (possibly empty) unions of tori. By Theorem 3.6 the natural Hermitian form on $V_r(\Sigma_1)$ and $V_r(\Sigma_2)$, are positive definite. Hence we have orthonormal bases $(\varphi_i)_i$ and $(\psi_j)_j$ orthonormal basis of $V_r(\Sigma_1)$ and $V_r(\Sigma_2)$, respectively. We can write:

$$RT_r(M_1) = \sum_i v_i\otimes\varphi_i,$$

where the v_i are vectors in $V_r(T)$ and

$$RT_r(M_2) = \sum_j w_j\otimes\psi_j,$$

where the w_j are vectors in $V_r(\overline{T})$. From this we get

$$TV_r(M_1) = ||RT_r(M_1)||^2 = \sum_i ||v_i||^2,$$

and likewise

$$TV_r(M_2) = ||RT_r(M_2)||^2 = \sum_j ||w_j||^2.$$

On the other hand, one has

$$\begin{aligned} TV_r(M) = ||RT_r(M)||^2 = ||\sum_{i,j}\langle v_i, w_j\rangle \varphi_i \otimes \psi_j||^2 = \sum_{i,j} |\langle v_i, w_j\rangle|^2 \\ \leqslant \sum_{i,j} ||v_i||^2||w_j||^2 = TV_r(M_1)TV_r(M_2), \end{aligned}$$

by the Cauchy-Schwarz inequality. □

As a special case of this theorem we get the following.

Corollary 5.3. *Let M' be a compact oriented 3-manifold with non-empty toroidal boundary and let M be a manifold obtained from M' by Dehn-filling some of the boundary components. Then*

$$TV_r(M) \leqslant TV_r(M'),$$

and thus

$$LTV(M) \leqslant LTV(M').$$

Proof. Suppose we are Dehn filling n components of $\partial M'$. The Dehn filling M is obtained from $M' \coprod_{i=1}^n V_i$ by gluing the boundary of each solid torus $V_i = D^2 \times S^1$ to a boundary component of M'. We can do the Dehn filling one component at a time. Thus Corollary 5.3 is an immediate consequence of Theorem 5.2, the multiplicativity of TV_r under disjoint union and the fact that

$$TV_r(D^2 \times S^1) = RT_r(S^2 \times S^1) = 1.$$

□

6. Bounds for Seifert manifolds

The previous section showed that the Turaev-Viro invariants are well behaved with respect to 3-manifold decompositions along tori. In this section we deal with large r asymptotic behavior of Turaev-Viro invariants for Seifert manifolds. We will use Corollary 5.3 to show the TV_r invariants of Seifert manifolds are at most polynomially growing. Our argument will be slightly different depending on whether the Seifert manifold has orientable or non-orientable base. The following lemma will help us reduce the latter to the former.

Lemma 6.1. *Let $\tilde{\Sigma}$ be a compact non-orientable surface. Then there is a simple closed curve γ on $\tilde{\Sigma}$ that is orientation reversing such that the surface $\Sigma = \tilde{\Sigma} \setminus \gamma$ obtained from $\tilde{\Sigma}$ by cutting along γ is orientable.*

Proof. Without loss of generality, we can assume $\tilde{\Sigma}$ is closed. Otherwise, we can fill the boundary components by disks to get a closed surface $\tilde{\Sigma}'$, and a simple closed curve for $\tilde{\Sigma}'$ that cuts it into an orientable surface. Isotopying this curve away from the filled in disks on $\tilde{\Sigma}'$ we will get a curve that satisfies the conclusion of the lemma for $\tilde{\Sigma}$.

Now, as closed non-orientable surfaces are characterized by their Euler characteristic which is at most 1, the surface $\tilde{\Sigma}$ is homeomorphic either to some $\mathbb{R}P^2\#(\#T^2)^p$ or to some $K^2\#(\#T^2)^p$ where K^2 is the Klein bottle and $p \geqslant 0$. As T^2 is orientable, it will then be sufficient to find such a path in $\mathbb{R}P^2$ and K^2. In $\mathbb{R}P^2$ such a path is given by the image of the diameter in S^2 by the cover map. If we view the Klein bottle K^2 as

$$K^2 \simeq S^1 \times [0,1]_{/(x,1)\sim(-x,0)},$$

then the path $S^1 \times \{0\}$ works. □

We are now ready to bound the Turaev-Viro invariants of Seifert fibered manifolds.

Theorem 6.2. *Let M be a compact orientable manifold that is Seifert fibered. Then there exist constants $A > 0$ and $N > 0$, depending on M, such that*

$$TV_r(M, e^{\frac{2i\pi}{r}}) \leqslant Ar^N.$$

Thus we have $LTV(M) \leq 0$.

Proof. We treat the case of orientable and non-orientable base separately.

Case 1: Let us first assume that the base of M is orientable. Drilling out exceptional fibers and possibly one regular fiber, we obtain a Seifert manifold which is a locally trivial fiber bundle over an oriented surface with non-empty boundary $\Sigma_{g,n}$. Here, $n \geqslant 1$ is the number of boundary components of $\Sigma_{g,n}$ and g is the genus. Because $H^2(\Sigma_{g,n}, \mathbb{Z}) = 0$, the Euler number of the fibration is zero and the S^1-fibration is globally trivial. In the end, M is a Dehn-filling of $\Sigma_{g,n} \times S^1$ for some oriented surface $\Sigma_{g,n}$ of genus g and $n \geqslant 1$ boundary components.

By Corollary 5.3, we have $TV_r(M) \leqslant TV_r(\Sigma_{g,n} \times [0,1])$. It remains to show that $TV_r(\Sigma_{g,n} \times S^1)$ is bounded by a polynomial. By Theorem 3.7, we have that

$$TV_r(\Sigma_{g,n} \times S^1) = RT_r(D(\Sigma_{g,n} \times S^1)) = RT_r(D(\Sigma_{g,n}) \times S^1).$$

The double surface $D(\Sigma_{g,n})$ is a closed orientable surface with $\chi(D(\Sigma_{g,n})) = 2\chi(\Sigma_{g,n}) = 4 - 4g - 2n$; thus it is the surface Σ_{2g+n-1} of genus $2g + n - 1$. So that we have

$$TV_r(\Sigma_{g,n} \times S^1) = RT_r(\Sigma_{2g+n-1} \times S^1) = \mathrm{Tr}(\rho_r(id_{\Sigma_{2g+n-1}})) = \dim(V_r(\Sigma_{2g+n-1})),$$

where ρ_r is the quantum representation of $V_r(\Sigma_{2g+n-1})$. The last quantity is a polynomial by Theorem 3.6(2).

Case 2: Assume that the base of M is a non-orientable surface $\tilde{\Sigma}$. By Lemma 6.1, we have a simple closed curve γ on $\tilde{\Sigma}$ such that $\Sigma = \tilde{\Sigma} \smallsetminus \gamma$ is orientable. One can assume that γ does not meet any exceptional fiber up to isotopying γ. The fibers of M corresponding to points on γ form an embedded Klein bottle K^2 in M. As γ is orientation reversing and does not meet exceptional fibers, a regular neighborhood of γ in $\tilde{\Sigma}$ will be a Möbius band that does not meet the exceptional fibers, and its total space by the Seifert fibration will be homeomorphic to the twisted I-bundle over the K^2.

$$K^2 \tilde{\times} I = [0,1] \times S^1 \times [-1,1]_{/(1,y,z)\sim(0,-y,-z)}.$$

The boundary of $K^2 \tilde{\times} I$ is a separating torus in M and, cutting M along this torus, one obtains on one side a Seifert manifold M' that fibers over the orientable surface Σ and $K^2 \tilde{\times} I$ on the other side.

By Theorem 5.2, $TV_r(M) \leqslant TV_r(M')TV_r(K^2 \tilde{\times} I)$.

We already know that $TV_r(M')$ is bounded by a polynomial so we only need to discuss $TV_r(K^2\tilde{\times}I)$. By Theorem 3.7 again, we have that $TV_r(K^2\tilde{\times}I) = RT_r(D(K^2\tilde{\times}I))$. The double of $K^2\tilde{\times}I$ is:

$$D(K^2\tilde{\times}I) = [0,1]\times S^1 \times S^1_{/(1,y,z)\sim(0,-y,-z)}.$$

This is the mapping torus of the elliptic involution s over T^2. Thus we have that

$$TV_r(K^2\tilde{\times}I) = \mathrm{Tr}(\rho_r(s)) = \dim(V_r(T^2)) = \frac{r-1}{2},$$

as the elliptic involution is in the kernel of all quantum representations ρ_r by Lemma 3.5. $\square$

7. Turaev-Viro invariants and simplicial volume

7.1. **A universal bound.** In this section we complete the proof of Theorem 1.1 and deduce some corollaries. First we note the following elementary properties of LTV and lTV.

Proposition 7.1. *LTV is subadditive under disjoint unions and connected sums of 3-manifolds while lTV is superadditive.*

Proof. By Theorems 3.4 and 3.7 the Turaev-Viro invariants are multiplicative under disjoint union. Thus we have $TV_r(M\coprod M') = TV_r(M)TV_r(M')$. and

$$\begin{aligned}\limsup_{r\to\infty}\frac{2\pi}{r}\log|TV_r(M\coprod M')| &= \limsup_{r\to\infty}\frac{2\pi}{r}\left(\log|TV_r(M)|+\log|TV_r(M')|\right)\\ &\leqslant LTV(M)+LTV(M')\end{aligned}$$

as the $\limsup$ operator is subadditive.
For a connected sum, we have

$$TV_r(M\# M') = \frac{TV_r(M)TV_r(M')}{TV_r(S^3)} = \eta_r^{-2}TV_r(M)TV_r(M').$$

But we have

$$\frac{\log|\eta_r|}{r} = \frac{\log|\frac{2\sin(\frac{2\pi}{r})}{\sqrt{r}}|}{r} = O(\frac{\log r}{r}).$$

So subadditivity of TV_r under connected sum follows again from the subadditivity of $\limsup$. The claims about lTV follow similarly as $\liminf$ is superadditive. $\square$

We are now ready to finish the proof of the main result of the paper that was stated as Theorem 1.1 in the introduction. We slightly restate the theorem.

Theorem 1.1. *There exists a universal constant C such that for any compact orientable 3-manifold M with empty or toroidal boundary we have*

$$LTV(M) \leqslant C||M||,$$

where the constant C is about 8.3581×10^9.

Proof. As both $LTV(M)$ is subadditive and $||M||$ additive under disjoint union and connected sum, it is enough to prove it for prime manifolds. As $TV_r(S^2\times S^1) = 1$ and $||S^2\times S^1|| = 0$, we can ignore $S^2\times S^1$ factors.

Next we treat the case of hyperbolic manifolds. By Theorem 2.4, if M is hyperbolic, there is a link L in M such that $M \smallsetminus L$ admits a triangulation with at most $C_2||M||$ tetrahedra, where C_2 is the universal constant defined in 2.3. By Corollary 5.3 and Proposition 1.2, we have

$$LTV(M) \leqslant LTV(M \smallsetminus L) \leqslant C_1 C_2 ||M||.$$

Setting $C = C_1 C_2$ we recall that C_1 has been estimated to be less than 1.101×10^9 in Subsection 2.3. Furthermore $C_1 = v_8 + 8\Lambda\left(\frac{\pi}{8}\right)$ which is about 7.5914. Thus the constant $C = C_1 C_2$ is about 8.3581×10^9.

By Theorem 6.2, $LTV(M) = 0$, if M is a Seifert fibered manifold. As M is a Dehn-filling of $\Sigma \times S^1$ for some surface with boundary Σ, its Gromov norm is 0 as $||\Sigma \times S^1|| = 0$ by Theorem 2.2. Thus the result is true in this case.

Now suppose that M is any compact, oriented 3-manifold that is closed or has toroidal boundary. By the Geometrization Theorem there is a collection of essential, disjointly embedded tori in $\mathcal{T} = \{T_1, \ldots, T_n\}$ in M, such that such that all the connected components of $M \smallsetminus \mathcal{T}$ are either Seifert fibered manifolds or hyperbolic. By the above discussion the result is true for each component of $M \smallsetminus \mathcal{T}$. The simplicial volume is additive over the components of $M \smallsetminus \mathcal{T}$ (Theorem 2.2).
By Proposition 7.1, LTV is subadditive over the components of $M \smallsetminus \mathcal{T}$. Applying Theorem 5.2 inductively we get

$$LTV(M) \leqslant LTM(M \smallsetminus \mathcal{T}) \leqslant C||M \smallsetminus \mathcal{T}|| = C||M||,$$

where $C = C_1 C_2$ is about 8.3581×10^9. This concludes the proof of Theorem 1.1. □

Next we discuss lower bounds for the Turaev-Viro invariants.

Corollary 7.2. *Let M, M' be compact, oriented 3-manifolds with empty or toroidal boundary and such that M is obtained by Dehn filling from M' and suppose that $lTV(M) > 0$. Then we have*

$$lTV(M') \geqslant lTV(M) > 0.$$

Proof. Since M is obtained by Dehn filling from M', Corollary 5.3 gives $TV_r(M) \leqslant TV_r(M')$, and thus $lTV(M')) \geqslant lTV(M)$. □

Corollary 7.2 applies in particular when M' is a knot complement in M; this application also gives the proof of Corollary 1.3.

Proof. (of Corollary 1.3) Let $K \subset S^3$ be the figure-8 knot or the Borromean rings. By [10, Corollary 5.2], for $M_K = S^3 \smallsetminus K$, we have $lTV(M_K) = v_3||M_K|| = vol(M_K) \geqslant 2v_3$, where the last inequality follows from the fact that the volume of the figure-8 knot complement is $2v_3$ while the volume of the Borromean rings complement is $2v_8$.
If L is a link in S^3 that contains K, then M_K is obtained by Dehn filling from $M_L = S^3 \smallsetminus L$. By Corollary 7.2 we have $lTV(M_L) \geqslant 2v_3$. □

Removing solid tori from a 3-manifold can also be thought as a special case of removing a Seifert fibered sub-manifold. This generalized operation also preserves exponential growth of the TV_r.

Corollary 7.3. *Let M be a compact oriented 3-manifold such that $TV_r(M)$ grows exponentially, that is $lTV(M) > 0$. Assume that S is a Seifert manifold embedded in M. Then*

$$lTV(M \smallsetminus S) \geqslant lTV(M) > 0.$$

Proof. Note that ∂S consists of $n \geqslant 1$ tori. By Theorem 5.2,

$$TV_r(M) \leqslant \left(\frac{r-1}{2}\right)^n TV_r(M \smallsetminus S) TV_r(S).$$

But by Theorem 6.2, there are constants $A > 0$ and N such that $TV_r(S) \leqslant A r^N$. Thus

$$TV_r(M \smallsetminus S) \geqslant A' r^{-N'} TV_r(M)$$

for some constants $A' > 0$ and N', and $lTV(M \smallsetminus S) \geqslant lTV(M) > 0$. □

7.2. **Sharper estimates and Dehn filling.** In this section, we will results of Futer, Kalfagianni and Purcell [15] to obtain much sharper relations between LTV and volumes of hyperbolic link complements. We will also address the question of the extent to which relations between Turaev-Viro invariants of hyperbolic volume survive under Dehn filling. To state our results we need some terminology that we will not define in detail. For definitions and more details the reader is referred to [14].

Over the years there has been a number of results about coarse relations between diagrammatic link invariants and the volume of hyperbolic links. See [14] and references therein. Using such results we obtain sharper bounds than the one of Theorem 1.1.

A *twist region* in a diagram is a portion of the diagram consisting of a maximal string of bigons arranged end-to-end, where maximal means there are no other bigons adjacent to the ends. We require twist regions to be alternating. The number of twist regions is the *twist number* of the diagram, and is denoted $\mathrm{tw}(D)$.

For a link L in S^3 with a diagram D with $tw(D)$ twist regions, the augmented link L' of L is obtained by adding a crossing circle around each twist region and replacing the twist region by two parallel strands. See, for example, [14, figure 2]. The complement of L can be obtained from the complement of L' by Dehn-filling along the boundary components corresponding to the crossing circles.

Theorem 7.4. *Let L be a link S^3, that admits a prime, twist reduced diagram*[1] *D with $\mathrm{tw}(D) > 1$, and such that each twist region has at least $n \geq 7$ crossings. Then L is hyperbolic and we have*

$$LTV(S^3 \smallsetminus L) \leqslant 10.4 \left(1 - \left(\frac{2\pi}{\sqrt{n^2+1}}\right)^2\right)^{-3/2} \mathrm{vol}(S^3 \smallsetminus L).$$

Proof. By a result of Agol and D. Thurston [20, Appendix]) the complement of the augmented link L' obtained from D, has a triangulation with at most $10(tw(D)-1)$ ideal tetrahedra. Thus by Theorem 1.2

$$LTV(S^3 \smallsetminus L') \leqslant (2.08) \cdot 10 v_8 \ (tw(D)-1) = (10.4) \cdot 2 v_8 \ (tw(D)-1).$$

[1] Every prime knot has prime twist reduced diagrams

The complement of the link L is obtained by Dehn-filling from $S^3\smallsetminus L'$. In fact, Futer and Purcell [16, Theorem 3.10] show that if each twist of D has at least n crossings, then all the filling slopes for the Dehn-filling from $S^3\smallsetminus L'$, to $S^3\smallsetminus L$, have length at least $\sqrt{n^2+1}$. Suppose that the diagram D of L has at least n crossings per twist region for some $n \geq 7$.

Then by [15, Theorem 1.2] and its proof, we have

$$2\ v_8(tw(D)-1)) \leqslant \left(1-\left(\frac{2\pi}{\sqrt{n^2+1}}\right)^2\right)^{-3/2} \mathrm{vol}(S^3\smallsetminus L).$$

By Corollary 5.3, we have

$$LTV(S^3\smallsetminus L) \leqslant LTV(S^3\smallsetminus L').$$

Now combining the three last inequalities we get the desired result.

□

Remark 7.5. Theorem 7.4 says that for most links we have $LTV(S^3\smallsetminus L) \leqslant 10.5\,\mathrm{vol}(S^3\smallsetminus L)$: Indeed, for links which at least n twists per twist region as above, for n large enough the inequality is satisfied. Then for every $B>0$, for links L that admit diagrams with $tw(D) \leq B$, for a generic choice of the number of twists in each twist region, the inequality is satisfied.

To continue, let M be a compact 3-manifold with toroidal boundary whose interior is hyperbolic, and let $T_1,\ldots,T_k$ be some components of ∂M. On each T_i, choose a slope s_i, such that the shortest length of any of the s_i is $\ell_{\min} > 2\pi$. Then the manifold $M(s_1,\ldots,s_k)$ obtained by Dehn filling along $s_1,\ldots,s_k$ is hyperbolic and [15, Theorem 1.1] gives a correlation between its volume and the volume of M.

The next result provides some information on how relations between Turaev-Viro invariants and hyperbolic volume behave under Dehn filling.

Corollary 7.6. *Let M be a compact 3-manifold with toroidal boundary whose interior is hyperbolic and let the notation be as above. Suppose that $LTV(M) = \mathrm{vol}(M)$. For $\ell_{\min} > 2\pi$ we have*

$$LTV(M(s_1,\ldots,s_k)) \ \leqslant\ B(\ell_{\min})\ \mathrm{vol}(M(s_1,\ldots,s_k)),$$

where $B(\ell_{\min})$ is a function that approaches 1 as $\ell_{\min} \to \infty$.

Proof. Since $\ell_{\min} \to \infty$, Theorem [15, Theorem 1.1] applies to give

$$\left(1-\left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M) \leqslant \mathrm{vol}(M(s_1,\ldots,s_k)).$$

Combining the last inequality with Corollary 5.3 we have

$$LTV(M(s_1,\ldots,s_k)) \ \leqslant LTV(M) = \mathrm{vol}(M) \leqslant \left(1-\left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{-3/2} \mathrm{vol}(M(s_1,\ldots,s_k)).$$

□

By [10, Theorem 1.6] if M is the complement of the figure-8 knot or the Borromean rings B we have $LTV(M) = \mathrm{vol}(M)$. Let K_n denote the double twist knot obtained by $1/n$-filling along each of two components of B. By Corollary 7.6, for any constant E arbitrarily close to 1, there is n_0 so that $LTV(S^3 \smallsetminus K_n) \leqslant\ E\ \mathrm{vol}((S^3 \smallsetminus K_n)$ whenever $n \geqslant n_0$.

8. Exact calculations of Gromov norm from Turaev-Viro invariants

In this section we give two examples of families of manifolds M such the growth rate of Turaev-Viro invariants detects the Gromov norm $||M||$ exactly. Both examples are derived as applications of the results in Section 5, 6 and 7. Both results provide partial verification of the following.

Conjecture 8.1. (Turaev-Viro invariants volume conjecture, [6]) *For every compact orientable 3-manifold M, with empty or toroidal boundary, we have*

$$LTV(M) = \limsup_{r\to\infty} \frac{2\pi}{r} \log |TV_r(M)| = v_3||M||,$$

where r runs over all odd integers.

A stronger version of conjecture 8.1 was first stated by Chen and Yang [6] for hyperbolic manifolds only and was supported by experimental evidence. The version above, which is the natural generalization of the conjecture in [6] was stated in [10] for links in S^3, where the authors and Yang also gave the first examples where the conjecture is verified. In particular, they proved it for knots in S^3 of simplicial volume zero. Here, as a corollary of Theorem 1.1 and Corollary 5.3 we generalize this later result as follows.

Corollary 8.2. *Suppose that M is a compact, orientable 3-manifold with $lTV(M) \geqslant 0$. Then, for any link $K \subset M$ with $||M \smallsetminus K|| = 0$, we have*

$$lTV(M) = LTV(M) = \lim_{r\to\infty} \frac{2\pi}{r} \log |TV_r(M \smallsetminus K)| = v_3||M \smallsetminus K|| = 0,$$

where r runs over all odd integers. That is Conjecture 8.1 holds for $M \smallsetminus K$.

In particular, the conclusion holds if $M = S^3$ or $\#(S^1 \times S^2)^k$.

Proof. By Theorem 1.1, $LTV(M \smallsetminus K) \leqslant 0$. By Corollary 5.3,

$$TV_r(M) \leqslant TV_r(M \smallsetminus K),$$

and thus $0 \leqslant lTV(M) \leqslant LTV(M \smallsetminus K) \leqslant 0$. Thus the conclusion follows.

The claim about S^3 or $\#(S^1 \times S^2)^k$ follows since, as it is easily seen by Theorem 3.4(2), we have $lTV(S^3), lTV(S^1 \times S^2) \geqslant 0$, and lTV is superadditive under connected sums. □

To describe our second family of examples, we introduce an operation we call *invertible cabling* that leaves both the Gromov norm and the growth rate of Turaev-Viro invariants unchanged.

Definition 8.3. A manifold S, with $||S|| = 0$ and with a distinguished torus boundary component T, is called an invertible cabling space if there is a Dehn-filling on some components of $\partial S \smallsetminus T$ that is homeomorphic to $T \times [0, 1]$.

A way to obtain invertible cabling spaces is to start with a link L in a solid torus such that L contains at least one copy of the core of the solid torus. One example of such a cabling space S is the complement in a solid torus of $p \geqslant 2$ parallel copies of the core.

Using Corollary 5.3 we also show the following:

Corollary 8.4. *Let M a 3-manifold with toroidal boundary for which Conjecture 8.1 holds and S be an invertible cabling space. Let M' be obtained by gluing a component of $\partial S \smallsetminus T$ to a component of ∂M. Then $LTV(M) = lTV(M) = lTV(M') = LTV(M')$, and thus Conjecture 8.1 holds for M.*

Proof. As there is a Dehn-filling on components of $\partial S \smallsetminus T$ that is homeomorphic to $T \times [0,1]$, M is a Dehn-filling of M' and $TV_r(M) \leqslant TV_r(M')$ by Corollary 5.3.
On the other hand, M' is obtained by gluing S to M along a torus. By Corollary 5.3 again,

$$TV_r(M') \leqslant TV_r(M) TV_r(S).$$

But as S has volume 0, by Theorem 6.2, we know that there exists constants A and N such that

$$TV_r(S) \leqslant A r^N.$$

On the other hand, we also have that $||M'|| = ||M \underset{T}{\cup} S|| \leqslant ||M|| + ||S|| = ||M||$ by Theorem 2.2, and also $||M|| \leqslant ||M'||$ as M is a Dehn-filling of M'. Thus M and M' have the same simplicial volume too. □

Corollary 8.4 applies in particular for M the complement of the figure-8 knot or to links with complement homeomorphic to the complement of the Borromean rings.

Appendix A. Proof of Lemma A.1

We prove Lemma A.1 from which we got the upper bound for $6j$-symbols in Section 4.2.

Lemma A.1. *The maximum of the function v is $\dfrac{v_8}{4}$ and the maximum of the function g is $8\Lambda\left(\dfrac{\pi}{8}\right)$*

Proof. The function v is differentiable and π-periodic in all variables, so such a maximum exists and is a critical point of v. Computing the partial derivatives $\dfrac{\partial v}{\partial \alpha}$, $\dfrac{\partial v}{\partial \beta}$, and $\dfrac{\partial v}{\partial \gamma}$, we see that (α, β, γ) is a critical point of v if and only if

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \Lambda'(\alpha+\beta+\gamma) \\ \Lambda'(\alpha+\beta-\gamma) \\ \Lambda'(\alpha+\gamma-\beta) \\ \Lambda'(\beta+\gamma-\alpha) \end{pmatrix} = 0.$$

The matrix $\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$ has rank 3 and kernel Vect $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, hence (α,β,γ) is a critical point of v if and only if

$$\Lambda'(\alpha+\beta+\gamma)=\Lambda'(\alpha+\beta-\gamma)=\Lambda'(\alpha+\gamma-\beta)=\Lambda'(\beta+\gamma-\alpha),$$

which given that $\Lambda'(x)=-\log(|2\sin x|)$ is equivalent to

$$|\sin(\alpha+\beta+\gamma)|=|\sin(\alpha+\beta-\gamma)|=|\sin(\alpha+\gamma-\beta)|=|\sin(\beta+\gamma-\alpha)|.$$

This means that the angles $\alpha+\beta+\gamma$, $\alpha+\beta-\gamma$, $\alpha+\gamma-\beta$ and $\beta+\gamma-\alpha$ are all equal or opposite mod π. Let us write $x=\alpha+\beta+\gamma=\pm(\alpha+\beta-\gamma)=\pm(\alpha+\gamma-\beta)=\pm(\beta+\gamma-\alpha)(\mathrm{mod}\ \pi)$.

Given that Λ is π periodic and odd, at such a critical point we have $v(\alpha,\beta,\gamma)=\frac{n}{2}\Lambda(x)$ where n is an even integer between -2 and 4. Moreover we have $v(\alpha,\beta,\gamma)=2\Lambda(x)$ if and only if

$$\alpha+\beta+\gamma=-(\alpha+\beta-\gamma)=-(\alpha+\gamma-\beta)=-(\beta+\gamma-\alpha)(\mathrm{mod}\ \pi).$$

This system is equivalent to $\alpha=0(\mathrm{mod}\ \dfrac{\pi}{4})$ and $\beta=\gamma=\alpha(\mathrm{mod}\ \dfrac{\pi}{2})$. We then see that the maximal value of v at such a critical point is $2\Lambda\left(\dfrac{\pi}{4}\right)=\dfrac{v_8}{4}$. This value is obtained for $\alpha=\beta=\gamma=\dfrac{3\pi}{4}$ or if two of the angles α,β,γ are equal to $\frac{\pi}{4}$ and the last one is $\frac{3\pi}{4}$.

For other critical points, where $v(\alpha,\beta,\gamma)=\dfrac{n}{2}\Lambda(x)$ with $|n|\leqslant 2$, the value of v is bounded by $\Lambda\left(\dfrac{\pi}{6}\right)$ as $\Lambda\left(\dfrac{\pi}{6}\right)=\dfrac{3}{2}\Lambda\left(\dfrac{\pi}{3}\right)=\dfrac{v_3}{2}$ is the maximum of Λ.

But we have that $v_3\simeq 1,01494\ldots<\dfrac{v_8}{2}=1.83419\ldots$. So the maximum of v is $\dfrac{v_8}{4}$.

Similarly, we see that $(Z,A_1,A_2,A_3,A_4,A_5,A_6)$ is a critical point of g if and only if

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \Lambda'(Z) \\ \Lambda'(Z-U_1) \\ \Lambda'(Z-U_2) \\ \Lambda'(Z-U_3) \\ \Lambda'(Z-U_4) \\ \Lambda'(V_1-Z) \\ \Lambda'(V_2-Z) \\ \Lambda'(V_3-Z) \end{pmatrix}=0.$$

The matrix $\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}$ has rank 7 and kernel Vect $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence $(Z, A_1, A_2, A_3, A_4, A_5, A_6)$ is a critical point of g if and only if

$$\begin{aligned}\Lambda'(Z) = \Lambda'(Z-U_1) = \Lambda'(Z-U_2) = \Lambda'(Z-U_3) = \Lambda'(Z-U_4)\\ = \Lambda'(V_1-Z) = \Lambda'(V_2-Z) = \Lambda'(V_3-Z),\end{aligned}$$

which is equivalent to

$$\begin{aligned}Z = \pm(Z-U_1) = \pm(Z-U_2) = \pm(Z-U_3) = \pm(Z-U_4)\\ = \pm(V_1-Z) = \pm(V_2-Z) = \pm(V_3-Z) (\text{mod } \pi).\end{aligned}$$

As above, the function being π-periodic and odd, at such a critical point we will have

$$g(Z, A_1, A_2, A_3, A_4, A_5, A_6) = -n\Lambda(Z),$$

with n an even integer between -6 and 8. Furthermore, $n = 8$ if and only if we have

$$Z = -(Z-U_i) = -(V_j - Z) \ (\text{mod } \pi).$$

From this we get $U_i = 2Z$ (mod π) and $V_j = 0$ (mod π). But, as

$$U_1 + U_2 + U_3 + U_4 = V_1 + V_2 + V_3,$$

we have that $8Z = 0$ (mod π).

Finally, as $\Lambda\left(\frac{\pi}{8}\right) \simeq 0.490936 > 0.457982 \simeq \Lambda\left(\frac{\pi}{4}\right)$ and

$$8\Lambda\left(\frac{\pi}{8}\right) \simeq 3.927488 > 3v_3 = 6\Lambda\left(\frac{\pi}{6}\right) \simeq 3,0448,$$

the maximum value of g is $8\Lambda\left(\frac{\pi}{8}\right)$.

Notice that this maximum is attained for $Z = \frac{7\pi}{8}$ and either all $A_i = \frac{\pi}{4}$, or all A_i are equal to $\frac{3\pi}{4}$ mod π, except two corresponding to opposite edges in the tetrahedron which are equal to $\frac{\pi}{4}$. □

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