Name ____________________________

PID ____________________________

Practice Final

MTH 310, Thursday June 28, 2018

Instructions: This exam is closed notes, closed books, no calculators and no electronic devices of any kind. There are five problems worth 20 points each. If a problem has multiple parts, it may be possible to solve a later part without solving the previous parts. Solutions should be written neatly and in a logically organized manner. Partial credit will be given if the student demonstrates an understanding of the problem and presents some steps leading to the solution. Correct answers with *no work* will be given *no credit*. The back sheets may be used as scratch paper but will not be graded for credit.

1	2	3	4	5	Total

Problem 1.

a. (10 points) Compute the remainder of $13^{2018} + 15^{310} - 5$ when divided by 14.

b. (10 points) Consider the polynomials $f(x) = x^2 + x + 2$ and $g(x) = x^2 - 1$ in $\mathbb{Z}_3[x]$. Find the gcd of $(f(x), g(x)) = d(x)$ and find $u(x), v(x) \in \mathbb{Z}_3[x]$ such that

$$f(x)u(x) + g(x)v(x) = d(x).$$

Problem 2. Let R be a ring and $s \in R$ be a fixed element. Define the set

$$S := \{srs : r \in R\}$$

a. (10 pts) Prove that S is a subring of R.

b. (10 pts) Suppose further that R is a ring with identity and that $s^2 = 1_R$. Prove that the map $f : R \to S$ defined by $f(r) = srs$ is a homomorphism.

Problem 3. Let $n \in \mathbb{Z}$ with $n > 0$.

a. (10 pts) Let $[a], [b] \in \mathbb{Z}_n$ and $I = \langle [a], [b] \rangle$ be the ideal generated by $[a]$ and $[b]$. Find $[d] \in \mathbb{Z}_n$ such that $I = \langle [d] \rangle$ is the principal ideal generated by $[d]$.

b. (10 pts) Suppose the gcd of $(d, n) = 1$. Prove that $\langle [d] \rangle = \mathbb{Z}_n$.

Problem 4. (20 pts.) Let F be a field and $f(x), g(x) \in F[x]$ both be of degree $\leq n$. Suppose that there are distinct elements $c_0, c_1, c_2, \cdots, c_n \in F$ such that $f(c_i) = g(c_i)$ for each i.

Prove that $f(x) = g(x)$ in $F[x]$.

Problem 5. Let I, J be ideals of a commutative ring R. Define the function

$$f : R \to (R/I) \times (R/J) \qquad \text{by} \qquad f(r) = (r + I, r + J).$$

You may use without proof that f is a homomorphism.

a. (10 pts) Prove that $\ker f = I \cap J$.

b. (10 pts) Prove that if $R = \{i + j : i \in I \text{ and } j \in J\}$ then f is surjective.
(Thus, by the First Isomorphism Theorem we have that $R/(I \cap J) \cong R/I \times R/J$.)