MTH 310: HW 6

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Due: June 25, 2018

1. (**Hungerford 5.3.5**) Verify that $\mathbb{Q}(\sqrt{3}) := \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. Then, show that $\mathbb{Q}(\sqrt{3})$ is isomorphic to $\mathbb{Q}[x]/\langle x^2 - 3\rangle$.
2. **(Hungerford 5.3.9)** Show that $\mathbb{Z}_2/\langle x^3 + x + 1\rangle$ is a field and contains all three roots of $x^3 + x + 1$.
3. **(Hungerford 6.1.6)** Show that the set of nonunits in $\mathbb{Z}_8$ is an ideal.
4. **(Hungerford 6.1.23)** Verify that $I = \{0, 3, 6, 9, 12\}$ is an ideal in $\mathbb{Z}_{15}$ and list all distinct cosets.
5. **(Hungerford 6.1.35)** Let $I \subset \mathbb{Z}$ be an ideal such that $\langle 3\rangle \subset I \subset \mathbb{Z}$. Prove that either $I = \langle 3\rangle$ or $I = \mathbb{Z}$.
6. Let $a \in \mathbb{R}$ and consider the evaluation homomorphism $\phi : \mathbb{R}[x] \to \mathbb{R}$ where $\phi(f(x)) = f(a)$. Find the kernel of ϕ.
7. **(Hungerford 6.2.12)** Let I be an ideal in a noncommutative ring R such that $ab - ba \in I$ for all $a, b \in R$. Prove that R/I is commutative.
8. **(Hungerford 6.2.21)** Use the First Isomorphism Theorem to show that $\mathbb{Z}_{20}/\langle 5\rangle$ is isomorphic to $\mathbb{Z}_5$.
9. **(Hungerford 6.3.5)** List all maximal ideals in $\mathbb{Z}_6$. Do the same in $\mathbb{Z}_{12}$.
10. **(Hungerford 6.3.13)**
    (a) Let $I \subset R$ be an ideal. Prove that $I \times I$ is an ideal in $R \times R$.
    (b) Prove that $(R \times R)/(I \times I)$ is isomorphic to $R/I \times R/I$. (*Hint*: Consider the function $f((a, b)) = (a + I, b + I)$.)