MTH 310: HW 5

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Due: June 18, 2018

1. Find all irreducible polynomials of degree 5 in $\mathbb{Z}_2[x]$. (*Hint*: There are six of them.)
2. **(Hungerford 4.3.21)** Find a non-constant polynomial in $\mathbb{Z}_9[x]$ that is a unit.
3. **(Hungerford 4.4.4)** For what value of k is $x+1$ a factor of $x^4+2x^3-3x^2+kx+1$ in $\mathbb{Z}_5[x]$.
4. **(Hungerford 4.4.19)** We say that $a \in F$ is a *multiple root* of $f(x) \in F[x]$ if $(x-a)^k$ is a factor of $f(x)$ for some $k \geq 2$. Prove that $a \in \mathbb{R}$ is a multiple root of $f(x) \in \mathbb{R}[x]$ if and only if a is a root of both $f(x)$ and $f'(x)$, where $f'(x)$ is the derivative of $f(x)$. You may use standard properties of the derivative like the product rule.
5. The Factor Theorem as proved in class has many corollaries to it. Read through Corollary 4.17, 4.18, 4.19, and 4.20 in the text and summarize the results.
6. **Rational Root Test:** Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ where $a_i \in \mathbb{Z}$ for each i. Let $r, s \in \mathbb{Z}$ with $r \neq 0$ and the gcd of $(r,s) = 1$. Show that if $\frac{r}{s}$ is a root, that is, $f(\frac{r}{s}) = 0$ then $r|a_0$ and $s|a_n$. [*Hint:* See Theorem 4.21 in Hungerford.]
7. **(Hungerford 5.1.6)** Let $a \in F$ and $f(x) \in F[x]$.

   (a) Show that $f(x) \equiv f(a) \mod (x-a)$.

   (b) Use (a) to show that $x^3+2 \equiv x^4+2x^2+1 \mod (x-2)$ in $\mathbb{Z}_5$.

   This problem shows that the congruence class of $f(x)$ modulo $x-a$ is determined only by the value of the polynomial when evaluated at a.
8. **(Hungerford 5.1.12)** Let $f(x), p(x) \in F[x]$. If $f(x)$ is relatively prime to $p(x)$, prove that there is a $g(x) \in F[x]$ such that $f(x)g(x) \equiv 1_F \mod p(x)$.
9. Write out the addition and multiplication tables for the ring $\mathbb{Z}_2[x]/(x^2+x)$. Is $\mathbb{Z}_2[x]/(x^2+x)$ a field?
10. In $\mathbb{Z}_2[x]/(x^3+x+1)$, find the the multiplicative inverse of $[x+1]$.
11. **(EC–worth .5% of final grade)** Let $p > 2$ be prime and consider the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by $f(x) = x^2$. Let $f(\mathbb{Z}_p)$ denote the image of f and find the cardinality $|f(\mathbb{Z}_p)|$. [*Hint*: $a \in f(\mathbb{Z}_p)$ if and only if the polynomial $x^2 - a$ is reducible in $\mathbb{Z}_p[x]$.]