MTH 310: HW 4

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Due: June 11, 2018

1. **(Hungerford 4.1.3)** List all the polynomials of degree 3 in $\mathbb{Z}_2[x]$.

   Solution. A polynomial of degree 3 has the form $ax^3 + bx^2 + cx + d$ for $a, b, c, d \in \mathbb{Z}_2$ and $a \neq [0]_2$. Therefore, $a = [1]$ and there are $2^3 = 8$ total degree 3 polynomials. They are

   $$x^3,\ x^3 + x^2,\ x^3 + x, x^3 + 1,\ x^3 + x^2 + x,\ x^3 + x^2 + 1,\ x^3 + x + 1,\ x^3 + x^2 + x + 1.$$

2. **(Hungerford 4.1.11)** Show that $1 + 3x$ is a unit in $\mathbb{Z}_9[x]$.

   Solution. Notice that $3 \cdot 6 = 9 \cdot 2$ so that $[3][6] = [0]$ in $\mathbb{Z}_9$. In otherwords, $[3]$ is a zero-divisor in $\mathbb{Z}_9$. It follows that

   $$(3x + 1)(6x + 1) = 18x + 9x + 1 = 1 \qquad \text{in} \qquad \mathbb{Z}_9.$$

   Thus, $3x + 1$ is a unit and its inverse is $(3x + 1)^{-1} = 6x + 1$.

3. **(Hungerford 4.1.16)** Let R be a commutative ring with identity and $a \in R$. If $1_R + ax$ is a unit in $R[x]$, show that $a^n = 0_R$ for some integer $n > 0$.

   Solution. Suppose that $1_R + ax$ is a unit in $R[x]$. Write its inverse as

   $$(1_R + ax)^{-1} = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

   and assume WLOG that $a_n \neq 0_R$. Multiplication gives

   $$\begin{aligned} 1_R &= (1_R + ax)(1_R + ax)^{-1} \\ &= (1_R + ax)(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) \\ &= a a_n x^{n+1} + (a a_{n-1} + a_n) x^{n-1} + \cdots + (a a_0 + a_1) x + a_0. \end{aligned}$$

   Equating coefficients we have that

   $$\begin{aligned} 1_R &= a_0 \\ 0_R &= a a_i + a_{i+1} \qquad \text{for} \quad 0 \leq i \leq n-1 \\ 0_R &= a a_n. \end{aligned}$$

   The above recursion relation is solve by $a_i = (-a)^i$ for $0 \leq i \leq n$. The last relation gives,

   $$0_R = a a_n = a(-a)^n = (-1)^n a^{n+1}.$$

   Since $(-1)^n$ is a unit in R we can apply cancellation. Therefore, $a^{n+1} = 0_R$.

4. **(Hungerford 4.1.20)** Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ be the derivative map defined by

   $$D(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}.$$

   Prove that D is not a homomorphism of rings.

   Solution. Recall that $D(x) = 1$. Notice that by the product rule

   $$D(x \cdot x) = x \cdot 1 + 1 \cdot x = 2x.$$

   It follows that $2x = D(x \cdot x) \neq D(x)D(x) = 1$. Thus, D is not a ring homomorphism.

5. **(Hungerford 4.2.4)** Let F be a field and $f(x), g(x) \in F[x]$. If $f(x)|g(x)$ and $g(x)|f(x)$ show that $f(x) = cg(x)$ for some nonzero $c \in F$.

   Solution. Since $f(x)$ and $g(x)$ are both divisors they are non-zero and have a non-negative degree.

   Let $p(x)$ and $q(x)$ be such that $f(x) = g(x)p(x)$ and $g(x) = f(x)q(x)$. Applying the degree formula to both we have

$$\deg f(x) = \deg g(x) + \deg p(x)$$
$$\deg g(x) = \deg f(x) + \deg q(x).$$

   Thus, $0 = \deg q(x) + \deg p(x)$. Since degree is non-negative we must have that $\deg q(x) = \deg p(x) = 0$ and thus $p(x) = c \in F$. It follows that $f(x) = cg(x)$.

6. **(Hungerford 4.2.5)**

   (a) Let $f(x) = x^4 + 3x^3 + 2x + 4$ and $g(x) = x^2 - 1$ in $\mathbb{Z}_5[x]$. Show that $g(x)|f(x)$.

   (b) Let $f(x) = x^4 + x + 1$ and $g(x) = x^2 + x + 1$ in $\mathbb{Z}_2[x]$. Adapt the Euclidean Algorithm for integers to find the gcd of $(f(x), g(x))$.

   Solution.

   (a) We can do long division and find that $x^4 + 3x^3 + 2x + 4 = (x^2 - 1)(x^2 + 3x + 1)$

$$\begin{array}{r}
x^2 + 3x + 1 \\
x^2 - 1 \overline{)\, x^4 + 3x^3 + 0x^2 + 2x + 4} \\
\underline{x^4 + 0x^3 - x^2} \qquad\qquad \\
3x^3 + x^2 + 2x + 4 \\
\underline{3x^3 + 0x^2 - 3x} \quad \\
x^2 + 0x + 4 \\
\underline{x^2 + 0x - 1} \\
0
\end{array}$$

   Thus, $g(x)|f(x)$.

   (b) We can do long division and find that $x^4 + x + 1 = (x^2 + x + 1)(x^2 + x) + 1$.

$$\begin{array}{r}
x^2 + x \\
x^2 + x + 1 \overline{)\, x^4 + 0x^3 + 0x^2 + x + 1} \\
\underline{x^4 + x^3 + x^2} \qquad\qquad \\
x^3 + x^2 + x + 1 \\
\underline{x^3 + x^2 + x} \quad \\
1
\end{array}$$

   Since $1 = (x^4 + x + 1) - (x^2 + x + 1)(x^2 + x)$ is the smallest degree monic polynomial linear combination of $f(x)$ and $g(x)$ we conclude that $(f(x), g(x)) = 1$.

7. **(Hungerford 4.2.15)** Let F be a field and $f(x), g(x), h(x) \in F[x]$. Prove that if $h(x)|f(x)$ and gcd of $(f(x), g(x)) = 1$ then gcd of $(h(x), g(x)) = 1$.

   Solution. Let $p(x), u(x), v(x) \in F[x]$ be such that $h(x)p(x) = f(x)$ and $1 = f(x)u(x) + g(x)v(x)$. By substitution, we have that $1 = h(x)p(x)u(x) + g(x)v(x)$. Since 1 is the smallest degree monic polynomial linear combination of $h(x)$ and $g(x)$ we conclude that $(h(x), p(x)) = 1$.

8. **(EC–worth .5% of final grade)** Let R be a commutative ring and let $f(x), g(x) \in R[x]$ with $f(x)$ nonzero. Prove that if $f(x)g(x) = 0_R$ then there exists $c \in R$ such that $cg(x) = 0_R$.

   [*Hint*: Let a be the leading coefficient of f. Show that there exist n such that $a^n g(x) = 0$.]