MTH 310: HW 3

Instructor: Matthew Cha

Due: May 30, 2018

1. **(Hungerford 3.1.6 b)** Let k be a fixed integer. Show that the set of multiples of k is a subring of $\mathbb{Z}$.

2. **(Hungerford 3.1.11 and 41)** Let $S \subset M_2(\mathbb{R})$ be the set of matrices of the form $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$.

   (a) Prove that S is a ring.

   (b) Show that $J = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is a *right indentity* (that is, $AJ = A$ for all $A \in S$). Show that J is not a left identity by finding a matrix $B \in S$ such that $JB \neq B$.

   (c) Prove that the matrix $\begin{pmatrix} x & x \\ y & y \end{pmatrix}$ is a right identity in S if and only if $x + y = 1$.

3. **(Hungerford 3.1.21)** Show that the subset $R := \{[0], [2], [4], [6], [8]\} \subset \mathbb{Z}_{10}$ is a subring of $\mathbb{Z}_{10}$ and that R is a ring with identity.

4. **(Hungerford 3.1.26)** Let $L = \{a \in \mathbb{R} : a > 0\}$. Define a new addition and multiplication on L by

$$a \oplus b = ab \qquad \text{and} \qquad a \otimes b = a^{\ln b}.$$

   Prove that L is a commutative ring with identity.

5. **(Hungerford 3.2.8)** Let R be a ring and $b \in R$ be fixed and define $T := \{rb : r \in R\}$. Prove that $T \subset R$ is a subring.

6. **(Hungerford 3.2.25)** Let $S \subset R$ be a subring and suppose R is an integral domain. Prove that S is an integral domain and that the identities are equal $1_S = 1_R$.

7. **(Hungerford 3.2.31)** A *Boolean ring* is a ring R with identity in which $x^2 = x$ for every $x \in R$. If R is a Boolean ring prove that R is commutative. [*Hint*: Expand $(a + b)^2$.]

8. **(Hungerford 3.3.9)** If $f : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that f is the identity map. [*Hint*: What is $f(1)$, $f(1 + 1)$, ...?]

9. **(Hungerford 3.3. 27 and 29)** If $g : R \to S$ and $f : S \to T$ are homomorphisms, show that $f \circ g : R \to T$ is a homomorphism. If f and g are isomorphisms, show that $f \circ g$ is an isomorphism.

10. **(Hungerford 3.3.41)** Let $m, n \in \mathbb{Z}$ be positive with gcd $(m, n) = 1$ and define the map $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $f([a]_{mn}) = ([a]_m, [a]_n)$.

    (a) Show that f is well-defined, that is, if $[a]_{mn} = [b]_{mn}$ then $[a]_m = [b]_m$ and $[a]_n = [b]_n$.

    (b) Prove that f is an isomorphism.

11. **(EC–worth .5% of final grade)** Let K be a field and suppose $K \subset L$ where L is a field. Then, L can be thought of as a vector space over K, for example, if $\mathbf{x}, \mathbf{y} \in L$ are vectors and $a, b \in K$ are scalars then $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$, $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{y}$, and $a(b\mathbf{x}) = (ab)\mathbf{x}$.

    Let K, L, M be fields with $K \subset L \subset M$. Suppose the dimension of M as a vector space over K is finite with dimension $d < \infty$. Prove that $d = mn$ where m is the dimension of M as a vector space over L and n is the dimension of L as a vector space over K.