MTH 310: HW 2

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Due: May 30, 2018

1. **(Hungerford 1.3.8)**
   (a) Verify that $x-1$ is a factor of x^n-1.
   (b) If n is a positive integer, prove that the prime factorization of $2^{2n}3^n-1$ includes 11 as one of the prime factors. [*Hint*: $(2^{2n}3^n=(2^23)^n)$.]
2. **(Hungerford 1.3.21)** If $c^2=ab$ and $(a,b)=1$, prove that a and b are perfect squares.
3. **(Hungerford 1.3.31)** If p is a positive prime, prove that $\sqrt{p}$ is irrational.
4. **(Hungerford 1.3.33)** Let $p>1$. If 2^p-1 is prime, prove that p is prime. [*Hint*: Prove the contrapositive: If p is composite, so is 2^p-1.]
5. **(Hungerford 2.1.3)** Every published book has a ten-digit ISBN-10 number that is usually of the form $x_1-x_2x_3x_4-x_5x_6x_7x_8x_9-x_{10}$, where each $0\leq x_i\leq 9$ is a single digit. Sometimes the last digit is the letter X, and should be treated as if it were the number 10. The first 9 digits identify the book. The last digit x_{10} is a *check digit*; it is chosen so that

   $$10x_1+9x_2+8x_3+7x_4+6x_5+5x_6+4x_7+3x_8+2x_9+x_{10}\equiv 0 \mod 11.$$

   If an error is made when scanning or keying the ISBN number into a computer the left side of the congruence will not be congruent to 0 modulo 11, and the number will be rejected as invalid. Which of the following are apparently valid ISBN numbers?

   (a) 3–540–90518–9 $\quad$ (b) 0–031–10559–5 $\quad$ (c) 0–385–49596–X.
6. **(Hungerford 2.1.8)** Prove that every odd integer is congruent to 1 modulo 4 or 3 modulo 4.
7. **(Hungerford 2.1.15)** If the greatest common divisor $(a,n)=1$, prove that there is an integer $b\in\mathbb{Z}$ such that $ab\equiv 1 \mod n$.
8. **(Hungerford 2.1.22)**
   (a) Give an example to show that the following statement is false: If $ab\equiv ac \mod n$ and $a\not\equiv 0 \mod n$, then $b\equiv c \mod n$.
   (b) Prove that the statement in part (a) is true whenever the gcd $(a,n)=1$.
9. **(Hungerford 2.2.11 and 15)** Solve the equation $x+x+x=[0]$ in $\mathbb{Z}_3$. (State the properties of modular arithmetic you are using in each step of your solution, see Theorem 2.7)

   Then, simplify the expression $([a]+[b])^3$ in $\mathbb{Z}_3$.
10. **(Hungerford 2.2.16)** Find all $[a]\in\mathbb{Z}_5$ for which the equation $[a]\cdot x=[1]$ has a solution.
11. **(Hungerford 2.3.2 and 6)** Find all zero divisors in (a) $\mathbb{Z}_7$ and (b) $\mathbb{Z}_9$.

    Next, prove that if n is composite then that there is at least one zero divisor in $\mathbb{Z}_n$.
12. **(Hungerford 2.3.10)** Prove that every nonzero element of $\mathbb{Z}_n$ is either a unit or a zero divisor, but not both.
13. **(Hungerford 2.3.17)** Prove that the product of two units in $\mathbb{Z}_n$ is also a unit.
14. **(EC–worth .5% of final grade)** Find all elements of the set $[2]_7\cap[3]_5$.