Name ______________________

PID ______________________

Final

MTH 310, Thursday June 28, 2018

Instructions: This exam is closed notes, closed books, no calculators and no electronic devices of any kind. There are five problems worth 20 points each. If a problem has multiple parts, it may be possible to solve a later part without solving the previous parts. Solutions should be written neatly and in a logically organized manner. Partial credit will be given if the student demonstrates an understanding of the problem and presents some steps leading to the solution. Correct answers with *no work* will be given *no credit.* The back sheets may be used as scratch paper but will not be graded for credit.

1	2	3	4	5	Total

Problem 1.

a. (10 points) Compute the remainder of 2^{310} when divided by 5. (*Hint*: $2^{310} = (2^2)^{155}$)

b. (10 points) Let $f : \mathbb{Z}_6 \to \mathbb{Z}_4$ be a homomorphism of rings with $f([1]_6) = [2]_4$. Compute $f([4]_6)$.

extra space for p.1

Problem 2. Let $p(x) = x^3 + 2x + 1$ in $\mathbb{Z}_3[x]$.

a. (6 pts) Show that $p(x)$ is irreducible in $\mathbb{Z}_3[x]$.

b. (7 pts) Find the inverse of $[x^2 + 1]$ in $\mathbb{Z}_3[x]/\langle p \rangle$.

c. (7 pts) How many elements are in the quotient ring $\mathbb{Z}_3[x]/\langle p \rangle$? Is $\mathbb{Z}_3[x]/\langle p \rangle$ a field?

extra space for p.2

Problem 3.

a. (6 pts) Show that $x^2 + 1$ has no roots in $\mathbb{Z}_7$.

b. (7 pts) Show that if $a \neq 0$ or $b \neq 0$ in $\mathbb{Z}_7$ then $a^2 + b^2 \neq 0$ in $\mathbb{Z}_7$.
(*Hint:* First, show that if $b \neq 0$ then $a^2 + b^2 = b^2((b^{-1}a)^2 + 1)$. Then, use part a.)

c. (7 pts) Consider the ring $\mathbb{Z}_7[i] := \{a + ib : a, b \in \mathbb{Z}_7\}$. Recall that $i^2 = -1$ and

$$\begin{aligned}(a + ib) + (c + id) &= (a + c) + (b + d),\\ (a + ib)(c + id) &= (ac - bd) + i(ad + bc).\end{aligned}$$

Prove that $\mathbb{Z}_7[i]$ is an integral domain. Is $\mathbb{Z}_7[i]$ a field?

extra space for p.3

Problem 4. Let $f : R \to S$ be a homomorphism of rings and $J \subset S$ be an ideal. Define the set

$$I = \{a \in R : f(a) \in J\} \subset R.$$

a. (6 pts) Show that $\ker f \subset I$.

b. (7 pts) Prove that I is a subring of R.

c. (7 pts) Prove that I is an ideal in R.

extra space for p.4

Problem 5. Let $\phi : \mathbb{Z}[x] \to \mathbb{Z}_3$ be defined by

$$\phi(a_0 + a_1 x + \cdots + a_n x^n) = [a_0]_3.$$

a. (10 pts) Prove that ϕ is a surjective ring homomorphism

b. (10 pts) Show that $\ker \phi = \langle 3, x \rangle$ is the ideal generated by 3 and x.
(Thus, by the First Isomorphism Theorem we have that $\mathbb{Z}[x]/\langle 3, x \rangle \cong \mathbb{Z}_3$.)

extra space for p.5

Extra Credit. (10 pts)

Let $n, p \in \mathbb{Z}$ be a positive, p be prime and $\langle p \rangle \subset \mathbb{Z}[x]$ denote the principal ideal generated by p. Suppose for $f(x), g(x), h(x), r(x), s(x) \in \mathbb{Z}[x]$ we have that

$$(f(x)r(x) + g(x)s(x)) + \langle p \rangle = 1 + \langle p \rangle$$

and

$$(f(x)g(x)) + \langle p \rangle = h(x) + \langle p \rangle.$$

Prove that there exist $F(x), G(x) \in \mathbb{Z}[x]$ such that the following hold

i. $F(x) + \langle p \rangle = f(x) + \langle p \rangle$,

ii. $G(x) + \langle p \rangle = g(x) + \langle p \rangle$,

iii. $F(x)G(x) + \langle p^n \rangle = h(x) + \langle p^n \rangle$.

extra space for e.c.