Name ______________________

PID ______________________

Exam 1

MTH 310, Thursday June 7, 2018

Instructions: This exam is closed notes, closed books, no calculators and no electronic devices of any kind. There are four problems worth 25 points each and one extrac credit worth 15 points. If a problem has multiple parts, it may be possible to solve a later part without solving the previous parts. Solutions should be written neatly and in a logically organized manner. Partial credit will be given if the student demonstrates an understanding of the problem and presents some steps leading to the solution. Correct answers with *no work* will be given *no credit.* The back sheets may be used as scratch paper but will not be graded for credit.

1	2	3	4	E	Total

Problem 1.

a. (15 points) Compute the remainder of 310^{2018} when divided by 3.

b. (10 points) Find the multiplicative inverse of $[8]_{31}$ in $\mathbb{Z}_{31}$.

extra space for p.1

Problem 2.

a. (15 pts) Let $a, b \in \mathbb{Z}$ not both zero and let gcd of $(a, b) = d$. Prove that if $a = dm$ and $b = dn$ for some $m, n \in \mathbb{Z}$ then the gcd of $(m, n) = 1$.

b. (10 pts) Let $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z} \setminus \{0\}$. Use part a. to show that $r = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with gcd of $(m, n) = 1$.

Problem 3. Let R and S be rings and $f : R \to S$ be a homomorphism of rings. Define the *kernel* of f as a subset of R by

$$\ker f := \{a \in R : f(a) = 0_S\} \subset R.$$

a. (10 pts) Prove that $\ker f$ is a subring of R.

b. (15 pts) Prove that f is an isomorphism if and only if f is surjective and $\ker f = \{0_R\}$.

extra space for p.3

Problem 4. Let $f : \mathbb{Z}_3 \to \mathbb{Z}_8$ be a ring homomorphism.

a. (5 pts) Let $[a]_8 \in \mathbb{Z}_8$. Show that if $[a]_8 + [a]_8 + [a]_8 = [0]_8$ then $[a]_8 = [0]_8$.

b. (10 pts) Use part a. to prove that $f([1]_3) = [0]_8$.

c. (10 pts) Use part b. to conclude that f must be the zero homomorphism, that is,

$$f(\ [b]_3\ ) = [0]_8 \qquad \text{for all} \quad [b] \in \mathbb{Z}_3.$$

(*Hint:* Write $[b]_3 = [b \cdot 1]_3 = [b]_3 \cdot [1]_3$.)

extra space for p.4

Extra Credit. (15 pts) Let p be prime and consider $\mathbb{Z}_p$. Define the set

$$S := \{(a, b) \in \mathbb{Z}_p \times \mathbb{Z}_p : a^2 = b^2 + 1 \text{ in } \mathbb{Z}_p\}.$$

Find the cardinality of the set S.

extra space for ec.