Name: Solutions ______________________ **Section**: ____________

Clear your desk of everything excepts pens, pencils and erasers. **Show all your work.**
If you have a question raise your hand and I will come to you.

1. Let $f(x) = \dfrac{x^2-4}{x^2-2x}$, $\quad f'(x) = \dfrac{-2}{x^2}$, $\quad f''(x) = \dfrac{4}{x^3}$.

(a) (1 point) Find the domain of $f(x)$.

The domain cannot have points where the denominator is 0.

$$0 = x^2 - 2x = x(x-2) \qquad \Longrightarrow \qquad x = 0 \text{ and } x = 2.$$

Domain: $\underline{(-\infty, 0) \cup (0, 2) \cup (2, \infty)}$.

(b) (1 point) Find all horizontal asymptotes of $f(x)$.

We want to compute the limits at infinity:

$$\lim_{x\to\infty} \frac{x^2-4}{x^2-2x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{1-\frac{2}{x}} = 1.$$

$$\lim_{x\to-\infty} \frac{x^2-4}{x^2-2x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1-\frac{4}{x^2}}{1-\frac{2}{x}} = 1.$$

There is one horizontal asymptote: $\underline{y = 1}$.

(c) (1 point) Find all vertical asymptotes of $f(x)$.

From part (a) that vertical asymptotes can occur at $x = 0$ and $x = 2$.

Factor the numerator: $x^2 - 2 = (x-2)(x+2)$.

$x = 2$ is a removable discontinuity and NOT a vertical asymptote.

$\underline{x = 0}$ is the vertical asymptote.

(d) (1 point) Show that $f(x)$ has NO local maxima or minima.

By the 1st derivative test, local max/min can occur only at critical points.

$f'(x)$ does not exists at $x = 0$, but that point is not in the domain.

Set $f'(x) = \frac{-2}{x^2} = 0$. There are no solutions.

We conclude there are no critical points. Therefore $f(x)$ cannot have local max/min.

(e) (0 points) Sketch the graph. Use a computer to compare with your sketch.

Continue on to back side

2. (2 points) Find the slant asymptote for $f(x) = \dfrac{x^2+x}{x-1}$.

Notice that the numerator has degree 2 and denominator has degree 1. They differ by 1, so this indicates that the function has a slant asymptote. To find the equation of the slant asymptote do long division"

$$\begin{array}{r} x+2 \\ x-1\,\overline{)\,x^2+x+0} \\ \underline{x^2-x} \\ 2x+0 \\ \underline{2x-2} \\ 2 \end{array}$$

It follows that

$$f(x) = \frac{x^2+x}{x-1} = (x+2) + \frac{2}{x-1}$$

We can immediately read off the slant asymptote: $\underline{y = x+2}$. Still need to compute the limit:

$$\lim_{x\to\infty} f(x) - (x+2) = \lim_{x\to\infty} \frac{2}{x-1} = 0.$$

3. (a) (2 points) Show that: $\lim_{x\to\infty} \sqrt{x^2+1} - x = 0$.

$$\begin{aligned} \lim_{x\to\infty} \sqrt{x^2+1} - x &= \lim_{x\to\infty} \sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} && \text{(multiply/divide by conjugate)} \\ &= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} && \text{(cancel the } x^2\text{)} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} && \text{(compare power on top/bot)} \\ &= 0. \end{aligned}$$

(b) (2 points) Show that: $\lim_{x\to-\infty} \sqrt{x^2+1} + x = 0$.

$$\begin{aligned} \lim_{x\to-\infty} \sqrt{x^2+1} + x &= \lim_{x\to-\infty} \sqrt{x^2+1} + x \cdot \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x} && \text{(multiply/divide by conjugate)} \\ &= \lim_{x\to-\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}-x} && \text{(cancel the } x^2\text{)} \\ &= \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}-x} && \text{(compare power on top/bot)} \\ &= 0. \end{aligned}$$

(c) (0 points) Using parts (a) and (b) above, find two slant asymptotes for $f(x) = \sqrt{x^2+1}$.

There are two slant asymptotes: $y = x$ and $y = -x$. Use a computer to graph $f(x)$.