Name: Solutions **Section**: 022

Clear your desk of everything excepts pens, pencils and erasers. **Show all your work.**
If you have a question raise your hand and I will come to you.

1. (2 points) **Multiple Choice. Circle the best answer. No partial credit available**

   Let $f(x) = \frac{|x-1|\cos(x-1)}{x-1}$. What is $\lim_{x\to 1^+} f(x)$?

   A. 1 ✓

   B. 0

   C. -1

   D. 5

   E. The limit does not exist (DNE).

   Recall that if $x > 1$ then $|x-1| = x-1$. Using the limit law for products we have that

$$\begin{aligned}\lim_{x\to 1^+} \frac{|x-1|\cos(x-1)}{x-1} &= \left(\lim_{x\to 1^+} \frac{|x-1|}{x-1}\right)\left(\lim_{x\to 1^+} \cos(x-1)\right)\\ &= \left(\lim_{x\to 1^+} \frac{x-1}{x-1}\right)\left(\lim_{x\to 1^+} \cos(x-1)\right)\\ &= (1)\cos(0) = 1.\end{aligned}$$

2. (2 points) **Multiple Choice. Circle the best answer. No partial credit available**

   Calculate: $\displaystyle\lim_{x\to -2} \frac{x^2+5x+6}{x^2+x-2}$

   A. $-\frac{1}{4}$

   B. $\frac{0}{0}$

   C. $-\frac{1}{3}$ ✓

   D. ∞

   E. $-\infty$

   Start by simplifying then look for cancellations.

$$\begin{aligned}\lim_{x\to -2} \frac{x^2+5x+6}{x^2+x-2} &= \lim_{x\to -2} \frac{(x+2)(x+3)}{(x+2)(x-1)}\\ &= \lim_{x\to -2} \frac{x+3}{x-1} = \frac{-2+3}{-2-1} = -\frac{1}{3}\end{aligned}$$

Extra Work Space.

Continue on to back side

3. (3 points) Use the Intermediate Value Theorem to show that the equation

$$x^2 = \frac{1}{x} + 1$$

has at least one solution in the interval $[1, \infty)$.

Step 0. (Rewrite the equation)

$$x^2 - \frac{1}{x} - 1 = 0 \qquad \text{multiply by } x \implies \qquad x^3 - 1 - x = 0.$$

Step 1. (Define $f(x)$ and N)

Let $f(x) = x^3 - x - 1$ and notice that $f(x)$ is continuous on the interval $[1, \infty)$.

Let $N = 0$. (We want to show that N is an intermediate value.)

Step 2. (Find a and b, so that N is an intermediate value.)

Let $a = 1$ then $f(1) = 1^3 - 1 - 1 = -1$, thus $f(1) < 0$.

Let $b = 2$ then $f(2) = 2^3 - 2 - 1 = 5$, thus $0 < f(2)$.

It follows that $f(1) < 0 < f(2)$ and therefore $N = 0$ is an intermediate value.

Step 3. (Use the IVT to finish)

We use the IVT to conclude that there is a solution c such that $1 < c < 2$ and $f(c) = 0$. This same c solves the original equation.

4. (3 points) Let $f(x) = \begin{cases} 3 - x^2 & \text{if } x \leq 0 \\ (x-1)^2 + a & \text{if } x > 0 \end{cases}$. Find a value for a so that f is continuous at $x = 0$.

Since the function is defined piecewise we should compute the left and right limits and set them equal:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3 - x^2 = 3.$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x-1)^2 + a = 1 + a.$$

For $f(x)$ to be continuous at $x = 0$ the right and left limits must be equal. Thus,

$$3 = 1 + a \qquad \implies \qquad \underline{a = 2}.$$

We finally conclude that $f(0) = 3 - (0)^2 = 0 = \lim_{x \to 0} f(x)$ and therefore by definition $f(x)$ is continuous at $x = 0$ if $a = 2$.