MTH 132: Sketch the graph

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October 29, 2018

Example. Let

$$f(x) = \frac{x^2+7}{x-4}, \qquad f'(x) = \frac{x^2-8x-7}{(x-4)^2}, \qquad f''(x) = \frac{46}{(x-4)^3}.$$

Sketch the graph of $f(x)$.

Solution.

1. Domain: $(-\infty, 4) \cup (4, \infty)$ since if $x = 4$ then the denominator is 0.

Critical points: Set $f'(x) = \frac{x^2-8x-7}{(x-4)^2} = 0$. The solutions are given by the quadratic formula: $x = (1/2)(8 \pm \sqrt{64+28}) = 4 \pm \sqrt{23} \approx -.8, 8.8$.

2. Local max/min: Test sign of f' in the intervals $(-\infty, 4-\sqrt{23})$, $(4-\sqrt{23}, 4)$, $(4, 4+\sqrt{23})$, and $(4+\sqrt{23}, \infty)$.

$$f'(-1) = \frac{2}{9} > 0, \quad f'(1) = \frac{-14}{9} < 0, \quad f'(5) = -22 < 0, \quad f'(10) = \frac{13}{36} > 0.$$

Increasing/decreasing test: $f(x)$ is increasing on $(4, 4+\sqrt{23})$ and $(4+\sqrt{23}, \infty)$ and $f(x)$ is decreasing on $(-\infty, 4-\sqrt{23})$ and $(4-\sqrt{23}, 4)$.

1st derivative test: $f(4-\sqrt{23})$ is local max and $f(4+\sqrt{23})$ is local min.

3. Concavity: Set $f''(x) = 0$, there are no solutions. Test the sign of f'' on $(-\infty, 4)$ and $(4, \infty)$.

$$f''(0) = \frac{46}{-4^3} < 0, \quad f''(5) = 46 > 0.$$

Concavity test: $f(x)$ is concave down on $(-\infty, 4)$ and $f(x)$ is concave up on $(4, \infty)$.

4. Asymptotes: There is a vertical asymptote at $x = 4$.

There is no horizontal asymptote.

There is a slant asymptote since the difference of order in the numerator and denominator is 1. Use long division to compute:

$$\begin{array}{r|l} & x+4 \\ \hline x-4 & x^2+0x+7 \\ & \underline{x^2-4x} \\ & \quad 4x+7 \\ & \quad \underline{4x-16} \\ & \qquad 23 \end{array}$$

Therefore, $f(x) = (x+4) + \dfrac{23}{x-4}$.

$$\lim_{x\to\infty} [f(x) - (x+4)] = \lim_{x\to\infty} \frac{23}{x-4} = 0.$$

$y = x + 4$ is the slant asymptote.

5.