Cosmetic crossings and genus-one knots

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Saturday, December 3, 2011

Joint work with S. Friedl, E. Kalfagianni and M. Powell
Crossing disks and crossing circles

Let $K$ be an oriented knot in $S^3$ and let $C$ be a crossing of $K$. 
Crossing disks and crossing circles

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- A **crossing disc** for \( K \) corresponding to \( C \) is an embedded disc \( D \subset S^3 \) such that \( K \) intersects \( \text{int}(D) \) twice, once for each branch of \( C \), with zero algebraic intersection number.

- \( L = \partial D \) is a **crossing circle** for \( K \) at \( C \).
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- $L = \partial D$ is a **crossing circle** for $K$ at $C$.

- A crossing change at $C$ is equivalent to performing $\varepsilon$-Dehn surgery on $L$, where $\varepsilon = -1$ if $C$ is a positive crossing and $\varepsilon = 1$ if $C$ is negative.
Nugatory crossing conjecture

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Known results
NCC is known to hold for:

- Unknot (Gabai, Scharleman-Thompson, 1989)
- 2-bridge knots (Torisu, 1999)
- Fibered knots (Kalfagianni, 2011)

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Our main goal
Find obstructions to cosmetic crossings for genus-one knots.
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- Let \( K', S' \) be obtained from \( K, S \) by \( \varepsilon \)-surgery on \( L \).
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Let $K', S'$ be obtained from $K, S$ by $\varepsilon$-surgery on $L$.

Note that if $M_{K \cup L}$ is reducible, then $C$ is a nugatory crossing.
Minimum-genus Seifert surfaces

Proposition
If \( K \cong K' \), then \( S \) and \( S' \) are minimum-genus Seifert surfaces for \( K \) and \( K' \), respectively, in \( S^3 \).

This follows from the following result of Gabai:

Gabai (1987)
Let \( M \) be a Haken manifold whose boundary is a nonempty union of tori. Let \( S \) be a Thurston norm minimizing surface representing an element of \( H_2(M, \partial M) \) and let \( P \) be a component of \( \partial M \) such that \( P \cap S = 0 \). Then with at most one exception (up to isotopy) \( S \) remains norm minimizing in each manifold \( M(a) \) obtained by filling \( M \) along an essential simple closed curve \( a \) in \( P \). In particular \( S \) remains incompressible in all but at most one manifold obtained by filling \( P \).
Crossing changes and Seifert surfaces

So a crossing change at $C$ which gives rise to an isotopic knot $K'$ corresponds to an embedded arc $\alpha$ on a minimum-genus Seifert surface $S$. 

If $\alpha$ is inessential, then $C$ is nugatory.

We want to consider a cosmetic crossing $C$ for a genus-one knot, so $\alpha$ will be essential (i.e. non-separating) on $S$. 
So a crossing change at $C$ which gives rise to an isotopic knot $K'$ corresponds to an embedded arc $\alpha$ on a minimum-genus Seifert surface $S$.

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Choosing a basis for $H_1S$

Let $\{a_1, a_2\}$ be a basis for $H_1S$ such that $a_1$ meets $\alpha$ and $a_2$ each once and $\alpha \cap a_2 = \emptyset$.

Note that $\{a_1, a_2\}$ also generates $H_1S'$.
Seifert matrices

We may choose appropriate orientations to obtain a Seifert matrix
\[
V = \begin{pmatrix} a & b \\ b + 1 & d \end{pmatrix}
\]
for \( S \). Then \( V' = \begin{pmatrix} a + \varepsilon & b \\ b + 1 & d \end{pmatrix} \) is a Seifert matrix for \( S' \).
Alexander polynomials

- $V$ and $V'$ give rise to the Alexander polynomials

\[ \Delta_K(t) \doteq \det(V - tV^T) = ad(1 - t)^2 - (b - (b + 1)t)((b + 1) - tb) \]

\[ \Delta_{K'}(t) \doteq (a + \varepsilon)d(1 - t)^2 - (b - (b + 1)t)((b + 1) - tb) \]

- Since $K \cong K'$, $d = \text{lk}(a_2, a_2) = 0$.
- So $K$ is algebraically slice. This gives us our first obstruction...
Obstructions

Main goal
Find obstructions to cosmetic crossings for genus-one knots.

Obstruction 1
If $K$ is a genus-one knot which admits a cosmetic crossing, then $K$ is algebraically slice. Hence $Δ_k(t) = f(t)f(t^{-1})$ for some $f(t) ∈ \mathbb{Z}[t]$ and $\det(K) = |Δ_k(-1)| = n^2$. 
Double branched covers

Let $Y_K$ be the double cover of $S^3$ branching over $K$ and define $Y_{K'}$ similarly.
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Then $V + V^T = \begin{pmatrix} 2a & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}$ is a presentation matrix for $H_1 Y_K$ and $V' + (V')^T = \begin{pmatrix} 2a + 2\varepsilon & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}$ is a presentation matrix for $H_1 Y_{K'}$. 
**$H_1 Y_K$ and $H_1 Y_{K'}$**

- $V + V^T = \begin{pmatrix} 2a & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}$ and $V' + (V')^T = \begin{pmatrix} 2a + 2\varepsilon & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}$.

- If $b = 0, -1$, then $H_1 Y_K = H_1 Y_{K'} = \{0\}$. 

This gives us our second obstruction...
Homology of double branched covers

\[ H_1 Y_K \text{ and } H_1 Y_{K'} \]

\( \begin{align*}
V + V^T &= \begin{pmatrix} 2a & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix} \text{ and } \\
V' + (V')^T &= \begin{pmatrix} 2a + 2\varepsilon & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}.
\end{align*} \)

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&\text{If } b = 0, -1, \text{ then } H_1 Y_K = H_1 Y_{K'} = \{0\}. \\
&\text{If } b \neq 0, -1, \text{ let } d = \gcd(2a, 2b + 1) \text{ and } d' = \gcd(2a + 2\varepsilon, 2b + 1).
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$H_1 Y_K$ and $H_1 Y_{K'}$

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- If $b \neq 0, -1$, let $d = \gcd(2a, 2b + 1)$ and $d' = \gcd(2a + 2\varepsilon, 2b + 1)$.

- Then $H_1 Y_K = \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d}}$ and $H_1 Y_{K'} = \mathbb{Z}_{d'} \oplus \mathbb{Z}_{\frac{(2b+1)^2}{d'}}$. 
$H_1 Y_K$ and $H_1 Y_{K'}$

1. $V + V^T = \begin{pmatrix} 2a & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}$ and $V' + (V')^T = \begin{pmatrix} 2a + 2\varepsilon & 2b + 1 \\ 2b + 1 & 0 \end{pmatrix}$.

2. If $b = 0, -1$, then $H_1 Y_K = H_1 Y_{K'} = \{0\}$.

3. If $b \neq 0, -1$, let $d = \gcd(2a, 2b + 1)$ and $d' = \gcd(2a + 2\varepsilon, 2b + 1)$.

4. Then $H_1 Y_K = \mathbb{Z}_d \oplus \mathbb{Z}_{(2b+1)^2}$ and $H_1 Y_{K'} = \mathbb{Z}_{d'} \oplus \mathbb{Z}_{(2b+1)^2}$.

5. $K \cong K' \Rightarrow H_1 Y_K = H_1 Y_{K'} \Rightarrow d = d' = 1 \Rightarrow H_1 Y_K = H_1 Y_{K'} = \mathbb{Z}_{(2b+1)^2}$.

6. This gives us our second obstruction...
Obstructions

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Find obstructions to cosmetic crossings for genus-one knots.

Obstruction 1
If $K$ is a genus-one knot which admits a cosmetic crossing, then $K$ is algebraically slice. Hence $\Delta_k(t) \div f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t]$ and $\det(K) = |\Delta_k(-1)| = n^2$.

Obstruction 2
If $K$ is a genus-one knot which admits a cosmetic crossing, then $H_1 Y_K$ is a finite cyclic group.
S-equivalence

If $K \cong K'$, then $V$ is S-equivalent to $V'$. 
**S-equivalence**

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**Question**

Can the matrices $\begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ and $\begin{pmatrix} a+1 & b \\ b+1 & 0 \end{pmatrix}$ be S-equivalent?
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be S-equivalent?

**Proposition**

For any $b > 4$ with $b \equiv 0$ or 2 (mod 3) there exists $a \in \mathbb{Z}$ such that
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are S-equivalent.

But there's hope yet...
Unique Seifert surfaces

If $K$ has a *unique* (up to isotopy) minimum-genus Seifert surface and $K \cong K'$, then $V$ and $V'$ must be congruent over $\mathbb{Z}$. 
Unique Seifert surfaces

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Proposition

If $\begin{pmatrix} a & b \\ b+1 & 0 \end{pmatrix}$ and $\begin{pmatrix} c & b \\ b+1 & 0 \end{pmatrix}$ are congruent over $\mathbb{Z}$, then there exists $n \in \mathbb{Z}$ such that $a + n(2b + 1) = c$.

If $c = a + 1$, this can only happen if $b = 0, -1$. 

Theorem

If $K$ is a genus-one knot with a unique minimum-genus Seifert surface and $K$ admits a cosmetic crossing, then $\Delta_k(t)$. 

Unique Seifert surfaces

If $K$ has a *unique* (up to isotopy) minimum-genus Seifert surface and $K \cong K'$, then $V$ and $V'$ must be *congruent* over $\mathbb{Z}$.

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**Theorem**

If $K$ is a genus-one knot with a unique minimum-genus Seifert surface and $K$ admits a cosmetic crossing, then $\Delta_k(t) \equiv 1$. 
Pretzel knots

Let $K = P(p, q, r)$ with $p$, $q$ and $r$ odd.

- $\det(P(p, q, r)) = |pq + qr + pr|$
- $g(K) \leq 1$
- $P(p, q, r)$ is algebraically slice if and only if $pq + qr + pr = -m^2$, for some odd $m \in \mathbb{Z}$
Pretzel knots

Corollary

A knot $P(p, q, r)$ with $p, q$ and $r$ odd does not admit cosmetic crossings if any of the following are true:

(a) $pq + qr + pr \neq -m^2$, for every odd $m \in \mathbb{Z}$
(b) $q + r = 0$ and $\gcd(p, q) \neq 1$
(c) $p + q = 0$ and $\gcd(q, r) \neq 1$
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Proof

(a) follows from Obstruction 1.
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Proof
(a) follows from Obstruction 1.
There is a genus one surface for \( P(p, q, r) \) which gives a Seifert matrix
\[
V = \frac{1}{2} \begin{pmatrix} p + q & q - 1 \\ q + 1 & q + r \end{pmatrix}
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and a presentation matrix \( \begin{pmatrix} p + q & q \\ q & q + r \end{pmatrix} \) for \( H_1 Y_{P(p,q,r)} \).
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\[ \begin{pmatrix} p + q & q \\ q & q + r \end{pmatrix} \]
for $H_1 Y_{P(p,q,r)}$. If $q + r = 0$ and $\gcd(p + q, q) \neq 1$, then $H_1 Y_{P(p,q,r)}$ is not cyclic, so (b) follows from Obstruction 2. A similar argument holds for (c).
Knots with at most 12 crossings

- There are 23 knots of genus one with at most 12 crossings.
Applications Knots with low crossing number

Knots with at most 12 crossings

▶ There are 23 knots of genus one with at most 12 crossings.
▶ Only four of these — $6_1$, $9_{46}$, $10_3$ and $11n_{139}$ — have a square determinant, and all four of these are algebraically slice.
Knots with at most 12 crossings

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- $6_1$ and $10_3$ are 2-bridge knots.
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- Only four of these — 6₁, 9₄₆, 1₀₃ and 1₁₁₁₃₉ — have a square determinant, and all four of these are algebraically slice.
- 6₁ and 1₀₃ are 2-bridge knots.
- 9₄₆ = P(3, 3, −3).
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- Only four of these — $6_1$, $9_{46}$, $10_3$ and $11_{139}$ — have a square determinant, and all four of these are algebraically slice.
- $6_1$ and $10_3$ are 2-bridge knots.
- $9_{46} = P(3,3,-3)$.
- This leaves $11_{139} = P(-5,3,-3)$. 
Knots with at most 12 crossings

$P(-5, 3, -3)$ has a genus-one Siefert surface with Seifert matrix $V = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$. So $H_1 Y_{P(-5,3,-3)} = \mathbb{Z}_9$. 

But $|\det(V)| = 2$, so we can make use of the following result of Trotter:

Trotter (1973)

Let $V$ be a Seifert matrix with $|\det(V)|$ a prime or 1. Then any matrix which is $S$-equivalent to $V$ is congruent to $V$ over $\mathbb{Z}$.

$V$ is congruent to $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ or $\begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}$ only if there exists $n \in \mathbb{Z}$ with $-1 + 5n = 0$ or $-1 + 5n = -2$, respectively.
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The End