Problem 1. [4+7+14=25 points] Your neighbor wants to enclose a 50 m$^2$ rectangular garden with a fence and then divide it into two equal rectangles, one for flowers and one for vegetables, using another fence (made of the same material) parallel to one of the sides of the outer rectangle. Her goal is to do this with as little fencing as possible.

(a) Draw a picture of this scenario and label all variables of constants.

\[ 2y \cdot x = 50 \text{ m}^2 \]

(b) Write the equation that needs to be optimized in terms of one variable and find the domain of this function in the context of this problem.

\[ F = 3x + 4y \]
\[ 2y \cdot x = 50 \Rightarrow y = \frac{25}{x} \]
\[ F = 3x + 100 \cdot \frac{25}{x} \]
\[ \text{Domain: } (0, \infty) \]

(c) What is the least amount of fence that your neighbor will need?

\[ F' = 3 - 100x^{-2} = 0 \]
\[ 3x^2 = 100 \]
\[ x = \sqrt{100/3} = \frac{10}{\sqrt{3}} \]
\[ F'' = -200x^{-3} \]
\[ F'' \left( \frac{10}{\sqrt{3}} \right) = 200 \left( \frac{\sqrt{3}}{10} \right)^3 > 0 \]
\[ F \text{ is } \min \text{ at } x = \frac{10}{\sqrt{3}} \text{ so this is } \min \]
\[ F \left( \frac{10}{\sqrt{3}} \right) = \frac{30}{\sqrt{3}} + 100 \cdot \frac{\sqrt{3}}{10} = 10\sqrt{3} + 10\sqrt{3} = 20\sqrt{3} \text{ m} \]
Problem 2. [10 points] Does the Mean Value Theorem apply to \( f(x) = x^{2/3} \) on the interval \([0, 1]\)? If so, explain why the two hypotheses of the Mean Value Theorem are satisfied and find the appropriate value \( c \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \) for the interval in question. If not, explain why it doesn’t apply.

\[
\begin{align*}
\text{Problem 3. } [3+3+6+4+6+4 = 26 \text{ points}] \\
\text{Let } f(x) = \frac{x^3 + x}{(x+1)^4}. \text{ One can easily verify that } \\
f'(x) = -\frac{(x-1)^3}{(x+1)^5} \text{ and } f''(x) = \frac{2(x-4)(x-1)^2}{(x+1)^6}. \text{ (You do not need to verify these these yourself.)}
\end{align*}
\]

(a) Find the domain of \( f(x) \).
\[
(x+1)^4 = 0 \quad \Rightarrow \quad x = -1 \\
\text{Domain} = (-\infty, -1) \cup (-1, \infty)
\]

(b) List all critical points of \( f(x) \).
\[
(x+1)^5 = 0 \quad \Rightarrow \quad x = -1 \quad \text{not in domain} \\
-(x-1)^3 = 0 \quad \Rightarrow \quad x = 1 \\
\boxed{c.p. \ x = 1}
\]

(c) Find all intervals where \( f(x) \) is increasing and all intervals where \( f(x) \) is decreasing.
\[
\begin{align*}
\text{Inc: } (-1, 1) \\
\text{Dec: } (-\infty, -1), (1, \infty)
\end{align*}
\]
(d) Find all local extrema of \( f(x) \) and determine which are local maxima and which are local minima.

\[
\begin{align*}
X = 1 & \text{ is loc. max,} \\
\text{No loc. min.}
\end{align*}
\]

(e) Find all intervals where \( f(x) \) is concave up and all intervals where \( f(x) \) is concave down.

\[
f'' = \frac{2(x-4)(x-1)^2}{(x+1)^6}
\]

\((x+1)^6 = 0 \Rightarrow x = -1 \text{ not in domain}\)

\[
2(x-4)(x-1)^2 = 0 \Rightarrow x = 4, 1
\]

\[
f''(x) = \begin{cases} - & x < 1 \\ + & x > 1 \end{cases}
\]

\[
f''(0) = \frac{2(-4)(1)}{1} < 0
\]

\[
f''(2) = \frac{2(-2)(1)}{3} < 0
\]

\[
f''(5) = \frac{2(1)(16)}{6^6} > 0
\]

\[
\text{c. \uparrow: (4, \infty)}
\]

\[
\text{c. \downarrow: (-\infty, 1), (1, \infty)}
\]

(f) Find all points of inflection of \( f(x) \).

\[
X = 4
\]
(g) Use this information to sketch a graph of \( f(x) \)

\[
f(x) = \frac{x^3 + x}{(x^2 + 2x + 1)^2}
\]

\[
\lim_{x \to -1^-} f(x) = \frac{-2}{\text{pos}} = -\infty \quad \text{vert. asympt.} \quad x = -1
\]

\[
\lim_{x \to -1^+} f(x) = \frac{-2}{\text{pos}} = -\infty
\]

\[
\lim_{x \to \infty} f(x) = 0
\]

\[
\lim_{x \to -\infty} f(x) = 0 \quad \text{hor. asympt.} \quad y = 0
\]

\[
f(0) = 0
\]

\[
f(1) = \frac{2}{16} = \frac{1}{8}
\]