Limits

* \( \lim_{x \to c} f(x) = L \) if and only if

\[
\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L
\]

1. \( \lim_{x \to 0} \left( \frac{\sqrt{2+x} - \sqrt{2}}{x} \right) \cdot \frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}} \)

\[
\lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x} + \sqrt{2})} = \lim_{x \to 0} \frac{x}{x(\sqrt{2+x} + \sqrt{2})} = \lim_{x \to 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
\]

2. \( \lim_{x \to 0} \left( \frac{\sin^2(5x)}{4x^2} \right) \)

\[
\lim_{x \to 0} \frac{\sin(5x) \cdot \sin(5x)}{5x \cdot 5x} \cdot \frac{1}{4x^2} \cdot \frac{25x^2}{x^2} = \frac{25}{4}
\]
Derivatives

Know:

- Power rule
- Quotient rule
- Chain rule (nested)
- 1st FTC

Know how to compute a deriv. directly from the definition

\[ \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or} \quad \lim_{z \to x} \frac{f(z)-f(x)}{z-x} \]

Find \( y' \) for the following:

3. \( y = (x^3 - \sin x) \cdot \cos (6x) \)

\[ y' = (x^3 - \sin x) (-\sin (6x)) \cdot 6 \]
\[ + \cos (6x) (3x^2 - \cos x) \]

4. \( y = \tan^4 (x^3 - x) = (\tan (x^3-x))^4 \)

\[ y' = 4 (\tan (x^3-x))^3 \cdot \sec^2 (x^3-x) \cdot (3x^2 - 1) \]

5. \( y = \int_{2}^{\sqrt{x}} t^8 (t+1)^{20} \, dt \)

\[ y' = (\sqrt{x})^8 (\sqrt{x} + 1)^{20} \cdot \frac{1}{2} x^{-1/2} \]
Implicit diff., tangent & normal lines

6. Given $x^2 + \sin y = xy^3 + 4$, find

- $\frac{dy}{dx}$
- Equations for the tangent & normal lines at the point $(2,0)$

1. $2x + \cos y \cdot y' = x \cdot 3y^2 \cdot y' + y^3$

\[
\cos y \cdot y' - 3xy^2 \cdot y' = y^3 - 2x
\]
\[
y'(\cos y - 3xy^2) = y^3 - 2x
\]
\[
y' = \frac{y^3 - 2x}{\cos y - 3xy^2}
\]

2. $y' = \frac{0 - 4}{1 - 0} = -4$

Tangent: $y = -4(x - 2)$

Normal: $y = \frac{1}{4}(x - 2)$
Positions functions; graphs of deriv.

* Know how to sketch a graph of the deriv. given a graph of the function and vice versa
* Know how position, velocity, speed & acceleration relate to each other

7. The graph of $f'(x)$ is shown below

![Graph of $f'(x)$]

- Where is the $f(x)$ increasing?  
  $(0, 2)$

- Is $f(2)$ a -local maximum - local minimum - inflection point - none of the above

- Where is $f(x)$ concave down?  $(1, 3)$
Related Rates

8. Coffee is draining from a conical filter at a rate of 16 in³/min. The filter stands point down & is 12 in. high & 8 in. wide.

- Draw a picture & write the radius of the coffee in the cone in terms of its height.

\[
\frac{r}{h} = \frac{4}{12} = \frac{1}{3} \quad r = \frac{1}{3} h
\]

- How fast is the level of the coffee falling when it is 2 in. deep? \(V = \frac{\pi}{3} r^2 h\)

Find \(\frac{dh}{dt}\) when \(h = 2\) in

\[
\frac{dV}{dt} = -16 \text{ in}^3/\text{min}
\]

\[
V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \left( \frac{1}{3} h \right)^2 \cdot h = \frac{\pi}{27} h^3
\]

\[
V = \frac{\pi}{27} h^3 \quad \Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{27} \cdot 2h^2 \frac{dh}{dt}
\]

\[
-\frac{3.6 \text{ in}^3/\text{min}}{\pi} = \frac{dh}{dt}
\]

\[-4 \text{ in/min} = \frac{dh}{dt}\]
MVT + Miscellaneous

* Linearization & differentials (end of Ch. 3)
* Newton’s method
* Initial value problems
  (e.g., \( s'' = 4 \cos(3t) \), \( s'(0) = 1 \), \( s(0) = \frac{5}{9} \))

9. For \( f(x) = \sqrt{x-4} \) explain whether \( f(x) \) satisfies the hypotheses of the Mean Value Theorem on the interval \([4, 8]\). If it does, find a point \( c \) that satisfies the conclusion of the Mean Value Theorem.

\[ \text{Domain: } [4, \infty) \text{ } \Rightarrow \text{ } f \text{ is cont. on domain} \]

* In particular, cont. on \([4, 8]\)

\( f'(x) = \frac{1}{2} (x-4)^{-1/2} = \frac{1}{2 \sqrt{x-4}} \)

\( f'(x) \) is defined on \((4, \infty)\)

* In particular, \( f \) is diff on \((4, 8)\)

Yes \( f(x) \) sat. hyp.

\[ f(8) - f(4) = f'(c) \]

\[ \frac{8-4}{2-0} = \frac{1}{2} \]

\[ \frac{4}{2} = \frac{1}{2} \]

\[ \frac{1}{2 \sqrt{x-4}} = \frac{1}{2} \]

\[ \sqrt{x-4} \Rightarrow x-4 = 1 \Rightarrow c = 5 \]
10. \[ \int x^3 \sec^2(x^4) \tan(x^4) \, dx \]
\[ u = \tan(x^4) \]
\[ du = \sec^2(x^4) \cdot 4x^3 \, dx \]
\[ \frac{1}{4} \int \tan(x^4) \cdot \sec^2(x^4) \cdot 4x^3 \, dx \]
\[ \frac{1}{4} \int u \, du = \frac{1}{4} \cdot \frac{1}{2} u^2 = \frac{1}{8} \tan^2(x^4) + C \]

11. \[ \int \frac{x}{\sqrt{x+1}} \, dx \]
\[ u = x + 1 \Rightarrow u - 1 = x \]
\[ du = dx \]
\[ \int (u-1)u^{-\frac{1}{2}} \, du = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du \]
\[ \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C = \frac{2}{3} (x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + C \]
12. \( \int_0^1 9(x^2+1) \sqrt{x^3+3x} \, dx \)

\[ u = x^3 + 3x \]
\[ du = 3x^2 + 3 \, dx \]
\[ = 3(x^2+1) \, dx \]

\[ u(1) = 1 + 3 = 4 \]
\[ u(0) = 0 \]

\[ 3 \int_0^1 \sqrt{x^3+3x} \cdot 3(x^2+1) \, dx \]

\[ = 3 \int_0^1 \frac{u^{1/2}}{3} \, du \]

\[ = 3 \left[ \frac{2}{3} u^{3/2} \right]_0^4 = 2u^{3/2} \bigg|_0^4 \]

\[ = 2 \cdot 8 - 2 \cdot 0 = 16 \]

13. \( \int_0^1 (x^2+1)(3x-2) \, dx \)

\[ = \int_0^1 (3x^3 - 2x^2 + 3x - 2) \, dx \]

\[ \frac{3}{4} x^4 - \frac{2}{3} x^3 + \frac{3}{2} x^2 - 2x \bigg|_0^1 \]

\[ = \left( \frac{3}{4} - \frac{2}{3} + \frac{3}{2} - 2 \right) - 0 \]

\[ = \frac{9 - 8 + 18 - 24}{12} = \frac{-5}{12} \]
Deriv. tests; asymptotes; sketching

14. Given \( f(x) = \frac{2 + x - x^2}{(x-1)^2} \), \( f'(x) = \frac{x - 5}{(x-1)^3} \)
\( f''(x) = \frac{2(7-x)}{(x-1)^4} \).

a) Find all asymptotes for \( f(x) \)

b) Find where \( f(x) \) is increasing and where \( f(x) \) is decreasing, as well as any local extrema.

c) Find where \( f(x) \) is concave up and where \( f(x) \) is concave down, as well as any inflection points.

d) Find the zeros of \( f(x) \) & sketch its graph

\[
\lim_{{x \to 1^-}} \left( \frac{2 + x - x^2}{(x-1)^2} \right) \to \frac{2}{(+infty)} \\
\lim_{{x \to 1^+}} \left( \frac{2 + x - x^2}{(x-1)^2} \right) \to \frac{2}{(+infty)} \to \infty
\]

\[
\lim_{{x \to \infty}} \left( \frac{-x^2 + x + 2}{x^2 - 2x + 1} \right) = -1
\]

vert. asymp. at \( x = 1 \)

hor. asymp. at \( y = -1 \)
b) \( f'(x) = \frac{x - 5}{(x - 1)^3} \Rightarrow f'(x) = 0 \Rightarrow x = 5 \) 
\( f'(x) \) DNE at \( x = 1 \)

\[ f' \quad \text{+} \quad - \quad + \]
\[ x \quad 1 \quad 5 \]

\( f'(0) = \frac{-5}{1} < 0 \)
\( f'(2) = \frac{-3}{1} < 0 \)
\( f'(8) = \frac{1}{5^3} > 0 \)

\( f \) inc: \((-\infty, 1) \cup (5, \infty)\)
\( f \) dec: \((1, 5)\)
loc. min. at \( x = 5 \)

c) \( f''(x) = \frac{2(7-x)}{(x-1)^4} \Rightarrow f'' = 0 \Rightarrow x = 7 \) 
\( f'' \) DNE at \( x = 1 \)

\[ f'' \quad \text{+} \quad - \quad + \]
\[ x \quad 1 \quad 7 \]

\( f''(0) = \frac{14}{+} > 0 \)
\( f''(2) = \frac{-10}{+} < 0 \)
\( f''(8) = \frac{-2}{+} < 0 \)

\( c, \uparrow: (-\infty, 1) \cup (1, 7) \)
\( c, \downarrow: (7, \infty) \)
inf. pt. at \( x = 7 \)

d) \( f(x) = \frac{2 + x - x^2}{(x-1)^2} \Rightarrow f(x) = 0 \)
\(- (x^2 - x - 2) = 0 \)
\(- (x - 2)(x + 1) = 0 \)
\( \Rightarrow x = 2, -1 \)
$\text{dec } (1, 5)$
$c. \uparrow (7, \infty)$
$\text{inf: } p + x = 7$
$\text{min } x = 5$

$\text{inc } x = 1$
$c. \uparrow$

$\text{dec } 1$
$c. \uparrow$
$c. \uparrow$
$c. \downarrow$
$1$
$5$
$7$
$\text{min}$
$\text{i.p.}$
Optimization

15. A rectangular box with volume $18 \text{ ft}^3$ is to be built with a square base and no top. The material for the bottom panel costs $\$2/\text{ft}^2$ and the material for the side panels costs $\$1.50/\text{ft}^2$. Find the minimum cost of such a box.

$$V = 18 \text{ ft}^3$$

$$C = 2x^2 + 4(1.5)xy$$

$$C = 2x^2 + 6xy$$

$$C = 2x^2 + 108x^{-1}$$

$$C' = 4x - 108x^{-2} = 0$$

$$4x^3 - 108 = 0$$

$$x^3 = \frac{108}{4} \Rightarrow x = 3$$

$$C'' = 4 - (-2)108x^{-3} = 4 + 216x^{-3} > 0 \Rightarrow C\text{ is concave up everywhere.}$$

$$\Rightarrow x = 3 \text{ is a min}$$

$$C(3) = 2(9) + \frac{108}{3} = 18 + 36 = \$54$$
Set up a Riemann sum approximation for $\int_{-2}^{2} x^3 \, dx$ by partitioning $[-2, 2]$ into 4 subintervals of equal length 4 using the right endpoint of each subinterval. Then evaluate this sum.

$$S = 1 \cdot (-1 + 0 + 1 + 8) = 8$$
### Areas

* Know how to find area bounded by 2 curves (e.g. \( y = 4 - x^2 \) and \( y = 3x \))

17. Find the total area of the shaded region.

\[ y = x^2 - 4x + 3 \]

\[ x^2 - 4x + 3 = (x-3)(x-1) \]

\[ A = \int_{0}^{3} (x^2 - 4x + 3) \, dx = \frac{1}{3} x^3 - 2x^2 + 3x \bigg|_{0}^{3} \]
\[ = \frac{1}{3} - 2 + 3 - 0 = \frac{4}{3} \]

\[ B \rightarrow \int_{1}^{3} (x^2 - 4x + 3) \, dx = \frac{1}{3} x^3 - 2x^2 + 3x \bigg|_{1}^{3} \]
\[ = 9 - 18 + 9 - \left( \frac{1}{3} - 2 + 3 \right) = -\frac{4}{3} \]

\[ B = -\frac{4}{3} \]

\[ C = \int_{3}^{5} (x^2 - 4x + 3) \, dx = \frac{1}{3} x^3 - 2x^2 + 3x \bigg|_{3}^{5} \]
\[ = \frac{125}{3} - 50 + 15 - \left( \frac{27}{3} - 27 + 9 \right) \]
\[ = \frac{125}{3} - 35 - \frac{1}{3} (25) = \frac{20}{3} \]

\[ A + B + C = \frac{4}{8} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3} \]