MATH 442: Homework 6  
Spring 2010

This is due the FRIDAY after spring break!

NOTE: For each homework assignment observe the following guidelines:

- Include a cover page.
- At the beginning of each problem, clearly state the problem (with assumptions) and what method or approach you intend to use to solve the problem.
- Clearly mark your answer.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the subplot command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.

The Wave Equation

1. Consider the boundary value problem:

\[ u_{tt} + u_{xxxx} + \delta u_t + ku = 0 \quad \text{in} \quad 0 < x < \pi \]
\[ u(0, t) = u(\pi, t) = 0 \]
\[ u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \]

where \( \delta, k > 0 \) are known constants, and \( \delta \) is small. This equation, which describes the vertical motion of a beam of length \( \pi \) with hinged ends, is called the beam equation. Use separation of variables to find the general solution of this equation. (\textbf{HINT:} you do not need to check the three cases \( \lambda < 0, \lambda = 0, \lambda > 0 \) separately. Only one case yields nonzero solution; you can simply focus on that case.)

2. Consider the following wave equation:

\[ u_{tt} = u_{xx} \quad \text{for} \quad 0 < x < 1, t > 0 \]
\[ u(0, t) = u(1, t) = 0 \]
\[ u(x, 0) = x^2(1 - x), \quad u_t(x, 0) = 0. \]

(a) Find the full solution. (\textbf{HINT:} the solution should not be in terms of infinite sums.)

(b) In MATLAB, plot the solution \( u(x, t) \) at \( t = 0 \) and at several later times.

(c) In MATLAB, plot \( u(0.5, t) \) versus \( t \).
3. Show that the solution to the initially unperturbed wave equation,

\[ u_{tt} = c^2 u_{xx} \]
\[ u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0 \]
\[ u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = g(x), \]

is

\[ u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\xi) \, d\xi, \]

where \( G(x) \) is the odd extension of \( g(x) \). (HINT: make use the separation of variables solution that we computed in class.)

What are \( F(x) \) and \( G(x) \)?

**Sturm-Liouville**

4. Consider the non-Sturm-Liouville differential equation

\[ d^2 \phi \over dx^2 + \alpha(x) \frac{d\phi}{dx} + \left[ \lambda \beta(x) + \gamma(x) \right] \phi = 0. \]

Multiply this equation by \( H(x) \). Determine \( H(x) \) such that the equation may be reduced to the standard Sturm-Liouville form:

\[ \frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + \left[ \lambda \sigma(x) + q(x) \right] \phi = 0. \]

Given \( \alpha(x) \), \( \beta(x) \), and \( \gamma(x) \), what are \( p(x) \), \( \sigma(x) \), and \( q(x) \).

5. Consider the eigenvalue problem

\[ x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} + \lambda \phi = 0 \quad \text{with} \quad \phi(1) = \phi(b) = 0. \]

(a) Use the result from the previous problem to put this in Sturm-Liouville form.

(b) Using the Rayleigh quotient, show that \( \lambda \geq 0 \).

(c) Solve this equation subject to the boundary conditions and determine the eigenvalues and eigenfunctions. Is \( \lambda = 0 \) an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the \( n \)th eigenfunction has \( n - 1 \) zeros in \( 1 < x < b \).

6. Consider

\[ c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + \alpha u, \]

where \( c \), \( \rho \), \( K_0 \), and \( \alpha \) are functions of \( x \), subject to

\[ u(0, t) = u(L, t) = 0 \]
\[ u(x, 0) = f(x). \]

Assume that the appropriate eigenfunctions are known.
(a) Using the Rayleigh quotient, show that the eigenvalues are positive if $\alpha < 0$.

(b) Solve the initial value problem.

(c) Discuss the limit as $t \to \infty$.

7. Consider the fourth-order linear differential operator:

\[ \mathcal{L} = \frac{d^4}{dx^4}. \]

(a) Show that $u\mathcal{L}(v) - v\mathcal{L}(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [u\mathcal{L}(v) - v\mathcal{L}(u)] \, dx$ in terms of the boundary data for any function $u$ and $v$.

(c) Show that $\int_0^1 [u\mathcal{L}(v) - v\mathcal{L}(u)] \, dx = 0$ if $u$ and $v$ are any two functions satisfying the boundary conditions

\[
\begin{align*}
\phi(0) &= \phi(1) = 0, \\
\phi'(0) &= \phi''(1) = 0.
\end{align*}
\]

(d) For the eigenvalue problem (using the boundary conditions from part (c))

\[
\frac{d^4\phi}{dx^4} + \lambda e^{x}\phi = 0,
\]

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

(e) Show that the eigenvalues in part (d) satisfy $\lambda \leq 0$. Is $\lambda = 0$ an eigenvalue?