Problem #6

Let \( y_1 \) and \( y_2 \) be solutions to the initial value problems:

\[
L(y_1) = 0, \quad y_1(5) = 0, \quad y_1'(5) = 2;
\]

\[
L(y_2) = 0, \quad y_2(5) = 0, \quad y_2'(5) = 4;
\]

where we introduced the linear operator \( L(y) = y'' + a(t)y' + b(t)y \).

1) Using the initial value problems for \( y_1 \) and \( y_2 \) find an initial value problem satisfied by the function \( y_c = y_1 + cy_2 \) where \( c \) is a constant. That is, find the function \( y_c \) and constants \( t_0, y_0, y_0' \) such that:

\[
L(y_c) = f_c(t), \quad y_c(t_0) = y_0, \quad y'_c(t_0) = y_0.'
\]

\[
L(y_c + cy_2) = L(y_1) + cL(y_2) \quad \text{because } L \text{ is linear operator}
\]

\[
= 0 + c(0) = 0
\]

\[
\Rightarrow f_c(t) = 0
\]

Since you are given 
\( y_1(5) = 0, \quad y_1'(5) = 2, \quad y_2(5) = 0, \quad y_2'(5) = 4 \)

\[
\Rightarrow t_0 = 5
\]

\[
y_c = y_1 + cy_2 \Rightarrow y_c(5) = y_1(5) + cy_2(5)
\]

\[
\Rightarrow y_c(5) = 0 + c(0) = 0 \quad \Rightarrow \quad y_c(5) = 0
\]

\[
y'_c = y'_1 + cy'_2 \Rightarrow y'_c(5) = y'_1(5) + cy'_2(5)
\]

\[
\Rightarrow y'_c(5) = 2 + 4c
\]

\[
\Rightarrow \boxed{y_c(5) = 2 + 4c}
\]
2) Find the value of the constant $c$ such that the function $y_c$, as defined in part (a), vanishes identically.

\[ y'_c(5) = 0 \Rightarrow 2 + 4c = 0 \Rightarrow c = \frac{-2}{4} \]

3) What result from section 2.1 in the lecture notes did you use in the calculation in part (b)?

Theorem 2.1.2
Problem #8

Let $y_1, y_2$ be fundamental solutions to a linear differential equation $L(y) = 0$.

Let $W = y_1 y'_2 - y'_1 y_2$ be their Wronskian functions and introduce

$$y_3 = ay_1 + by_2, \quad y_4 = cy_1 + dy_2,$$

where $a, b, c, d$ are constants.

1) Find the Wronskian function $\tilde{W} = y_3 y'_4 - y'_3 y_4$ as a function of $W$. That is, find $F$ such that $\tilde{W} = F(W)$.

$$\tilde{W} = y_3 y'_4 - y'_3 y_4$$

$$= (ay_1 + by_2)(cy'_1 + dy'_2) - (cy_1 + dy_2)(ay'_1 + by'_2)$$

$$= ac(y_1 y'_2 - y_2 y'_1) + bc(y'_1 y_2 - y_1 y'_2)$$

$$= acW + bcW$$

$$= (ac + bc)W$$

$$\Rightarrow \tilde{W} = \sqrt{F(W) = (ac + bc)W}$$

2) Given the constants $a, b, c$, with $a \neq 0$, find constant $d$ such that the functions $y_3, y_4$ are not fundamental solutions of $L(y) = 0$.

\[ y_3, y_4 \text{ are not fundamental} \Rightarrow \tilde{W} = 0 \]

\[ \Rightarrow (ac + bc)W = 0 \quad \text{since } W \neq 0 \]

\[ \Rightarrow ac + bc = 0 \Rightarrow ad = bc \]

\[ \Rightarrow \boxed{d = \frac{bc}{a}} \]