problem(6) Consider the IVP,

\[-2 \sin(3t) y' - 10 \cos(t) y' + 4 t^4 e^{2y} y' - 2y y' - 6y \cos(3t) + 10y \sin(t) + 8t^3 e^{-10t} = 0\]

\[y(0) = 3\]

1) Express the differential equation in the form \(N y' + M\), and then check if it is exact or not.

\[(-2 \sin(3t) - 10 \cos(t) + 4 t^4 e^{-2y}) y' + (-6y \cos(3t) + 10y \sin(t) + 8t^3 e^{-10t}) = 0\]

\[N = -2 \sin(3t) - 10 \cos(t) + 4 t^4 e^{-2y} - 2y\]

\[M = -6y \cos(3t) + 10y \sin(t) + 8t^3 e^{-10t}\]

\[\frac{\partial N}{\partial y} = -6 \cos(3t) + 10 \sin(t) + 16 t^3 e^{-2y}\]

\[\frac{\partial M}{\partial t} = -6 \cos(3t) + 10 \sin(t) + 16 t^3 e^{-2y}\]

\[\Rightarrow \frac{\partial N}{\partial y} = \frac{\partial M}{\partial t} \Rightarrow \text{Exact}.

2) Find the potential function \(\Phi(t, y) = C\)

i) Set \(\Phi = \int M dy + g(t)\)

\[\Phi = \int (-6y \cos(3t) + 10y \sin(t) + 8 t^3 e^{-10t}) \, dy + g(t)\]

\[\Phi = -6y \cos(3t) y - 10y \sin(t) y + 2t^4 e^{-10t} - y^2 + g(t)\]

ii) Take \(\frac{\partial \Phi}{\partial t}\) for both sides for \(\Phi\) in step(i) and use

\[\frac{\partial \Phi}{\partial t} = M\]

\[\frac{\partial \Phi}{\partial t} = -6y \cos(3t) + 10y \sin(t) + 8 t^3 e^{-10t} + g'(t)\]

\[\Rightarrow -6y \cos(3t) + 10y \sin(t) + 8 t^3 e^{-10t} = -6y \cos(3t) + 10y \sin(t) + 8 t^3 e^{-10t} + g'(t)\]

\[\Rightarrow g'(t) = -10t^4\]

\[\Rightarrow \boxed{g(t) = -2 t^5}\]
Therefore,

\[ \tilde{\psi}(t,y) = -2\sin(3t)y - 10\cos(t)y + 2t^4e^y - y^2 - 2t^5 \]

3) Find the unique potential function \( \Psi(t,y) \) that satisfies

\[ \Psi(t,y) = 0 \quad \text{for } y(0) = 3 \]

\[ \tilde{\Psi}(0,3) = -39 \]

Then, define \( \Psi(t,y) = \tilde{\Psi} + 39 \)

\[ \Psi(t,y) = -2y\sin(3t) - 10y\cos(t) + 2t^4e^y - y^2 - 2t^5 + 39 \]
Consider the differential equation:

\[ 14 \cos(7t) y' - 2 \sin(7t) + 40 e^{(y+t)} \sin(8t) - 5 e^{(y+t)} \cos(8t) y' + 8 \cos(8t) e^t = 0 \]

1) Express the differential equation in the form \( Ny' + M = 0 \) and then find \( \frac{\partial N}{\partial t} \) and \( \frac{\partial M}{\partial y} \):

\[ N = 14 \cos(7t) - 5 e^{(y+t)} \cos(8t) \]
\[ M = -2 \sin(7t) + 40 e^{(y+t)} \sin(8t) + 8 \cos(8t) e^t \]
\[ \frac{\partial N}{\partial t} = -5 e^{(y+t)} \cos(8t) + 40 e^{(y+t)} \sin(8t) \]
\[ \frac{\partial M}{\partial y} = -14 \cos(7t) + 40 e^{(y+t)} \sin(8t) \]

\[ \frac{\partial N}{\partial t} \neq \frac{\partial M}{\partial y} \implies \text{Not Exact} \]

2) Find the integrating factor \( \mu(t) \) such that satisfying \( M(0) = 1 \):

\[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} = -14 \cos(7t) + 5 e^{(y+t)} \cos(8t) \]
\[ \frac{2M}{\partial y} - \frac{\partial N}{\partial t} = \frac{-14 \cos(7t) + 5 e^{(y+t)} \cos(8t)}{14 \cos(7t) - 5 e^{(y+t)} \cos(8t)} = -1 \]

\[ \mu(t) = e^{\int \frac{2M}{\partial y} - \frac{\partial N}{\partial t} \, dt} = e^{\int (-1) \, dt} = e^{-t} \]

\[ \mu(t) = e^{-t} \]
3) Compute the functions $\tilde{N} = MN$ and $\tilde{M} = MM$ and then verify that $\tilde{N}y + \tilde{M} = 0$ is indeed exact.

$$\tilde{N} = MN = e^{-t}(14 \cos(7t) - 5 e^y \cos(8t))$$

$$\tilde{N} = 14 e^{-t} \cos(7t) - 5 e^y \cos(8t)$$

$$\tilde{M} = MM = e^{-t}(-2 \sin(7t) + 40 e^y \sin(8t) + 8 \cos(8t))$$

$$\tilde{M} = -2 e^{-t} \sin(7t) + 40 e^y \sin(8t) + 8 \cos(8t)$$

$$\frac{\partial \tilde{N}}{\partial t} = -14 e^{-t} \cos(7t) + 40 e^y \sin(8t)$$

$$\frac{\partial \tilde{M}}{\partial y} = -14 e^{-t} \cos(7t) + 40 e^y \sin(8t)$$

$$\Rightarrow \frac{\partial \tilde{N}}{\partial y} = \frac{\partial \tilde{M}}{\partial y} \Rightarrow \text{Exact}.$$  

4) Find the potential function $\tilde{V}(t,y)$

i) set $\tilde{V} = \int \tilde{N} \, dy + g(t)$

$$\tilde{V} = \int (14 e^{-t} \cos(7t) - 5 e^y \cos(8t)) \, dy + g(t)$$

$$\tilde{V} = 2 e^{-t} \sin(7t) - 5 e^y \cos(8t) + g(t)$$

ii) take $\frac{\partial \tilde{V}}{\partial t}$ for $\tilde{V}$ in step (i) and use $\frac{\partial \tilde{V}}{\partial t} = \tilde{M}$

$$\frac{\partial \tilde{V}}{\partial t} = -2 e^{-t} \sin(7t) + 40 e^y \sin(8t) + g'(t)$$

$$\Rightarrow -2 e^{-t} \sin(7t) + 40 e^y \sin(8t) + 8 \cos(8t) = -2 e^{-t} \sin(7t) + 40 e^y \sin(8t) + g'$$

$$\Rightarrow g'(t) = 8 \cos(8t) \Rightarrow g(t) = 8 \sin(8t)$$

$$\Rightarrow \tilde{V}(t,y) = 2 e^{-t} \sin(7t) - 5 e^y \cos(8t) + \sin(8t)$$