

### Table 1: Life Table

<table>
<thead>
<tr>
<th>Age</th>
<th>$l_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>52</td>
<td>89948</td>
</tr>
<tr>
<td>53</td>
<td>89089</td>
</tr>
<tr>
<td>54</td>
<td>88176</td>
</tr>
<tr>
<td>55</td>
<td>87123</td>
</tr>
</tbody>
</table>

**Michigan State University**
Department of Mathematics
STT 455 Fall 14
TEST 1
Name (Print): ____________________________
Name (Sign): ____________________________

Each Question is worth 10 points. Please read each question carefully and show all work to receive any credit.

- For Questions 1.) and 2.), consider a model of survival of animals in a colony with maximum age of 90 and $S_0(x) = \frac{9000-10x-x^2}{9000}$.

- For Question 3.) and 4.), consider an animal colony where mortality is modeled as a constant $\mu$.

- For Question 5.), assume that
  \[
  3p_{x+1} = 0.9500 \\
  p_x = 0.9900 \\
  p_{x+1} = 0.9850 \\
  q_{x+3} = 0.0200
  \]  
  (1)

- For Question 6.), assume that Gompertz’ Law applies with
  \[
  \mu_{30} = 0.000130 \\
  \mu_{50} = 0.000344
  \]  
  (2)

- For Question 7.), assume constant force of mortality and that $p_{40} = 0.95$

- For Question 8.), assume the life table given in Table 1.
1. Compute $p_{50}$

ANSWER:

\[
S_x(t) = \frac{S_0(x + t)}{S_0(x)}
\]

\[
\therefore S_{50}(1) = p_{50} = \frac{S_0(51)}{S_0(50)}
\]

\[
= \frac{9000 - 10 \cdot 51 - 51^2}{9000 - 10 \cdot 50 - 50^2}
\]

\[
= \frac{9000 - 10 \cdot 51 - 51^2}{9000 - 10 \cdot 50 - 50^2}
\]

\[
= \frac{5889}{6000} = 0.9815
\]

2. Compute $\mu_{50}$

ANSWER:

\[
\mu_x = -\frac{d}{dx} \frac{S_0(x)}{S_0(x)} = \frac{10 + 2x}{9000 - 10x - x^2}
\]

\[
\therefore \mu_{50} = \frac{10 + 2 \cdot 50}{9000 - 10 \cdot 50 - 50^2}
\]

\[
= \frac{110}{6000} = 0.01833
\]
3. Compute the curtate expectation of life for an animal in this colony

ANSWER:

Since the force of mortality is constant, we know that \( \forall x \in \mathbb{R}_+ \)

\[
t_p x = e^{- \int_{x}^{x+t} \mu s \, ds} = e^{- \int_{x}^{x+t} \mu ds} = e^{-\mu t} \quad (5)
\]

It follows that

\[
e_x = \lim_{k \to \infty} k p_x = \sum_{k=1}^{\infty} e^{-\mu k} = \frac{1}{1 - e^{-\mu}} - 1 = \frac{e^{-\mu}}{1 - e^{-\mu}} = \frac{1}{e^{\mu} - 1} \quad (6)
\]

4. Compute the complete expectation of life for an animal in this colony

ANSWER:

Since the force of mortality is constant, we know that \( \forall x \in \mathbb{R}_+ \)

\[
t_p x = e^{- \int_{x}^{x+t} \mu s \, ds} = e^{- \int_{x}^{x+t} \mu ds} = e^{-\mu t} \quad (7)
\]

It follows that

\[
\hat{e}_x = \int_0^\infty t \, p_x \, dt = \int_0^\infty e^{-\mu t} \, dt = \frac{1}{\mu} \quad (8)
\]
5. Calculate $2p_{x+1}$.

**ANSWER:**

We have $3p_{x+1} = 2p_{x+1} \cdot p_{x+3} = 2p_{x+1} \cdot p_{x+3}$, and so

$$2p_{x+1} = \frac{3p_{x+1}}{p_{x+3}} = \frac{3p_{x+1}}{1-q_{x+3}} = \frac{0.95}{0.98}, \quad (9)$$

6. Compute $\mu_{40}$

**ANSWER:**

We have

$$\mu_{30} = Bc^{30}$$
$$\mu_{50} = Bc^{50}$$

$$\Rightarrow c = \left( \frac{\mu_{50}}{\mu_{30}} \right)^{\frac{1}{2}} \quad (10)$$

$$\Rightarrow B = \frac{\mu_{30}}{\left( \frac{\mu_{50}}{\mu_{30}} \right)^{30}} = \left( \frac{\mu_{30}}{\mu_{50}} \right)^{\frac{5}{2}}$$

and so

$$\mu_{40} = Bc^{40} = \left( \frac{\mu_{30}}{\mu_{50}} \right)^{\frac{5}{2}} \cdot \left( \frac{\mu_{50}}{\mu_{30}} \right)^{\frac{4}{2}}$$
$$= \sqrt{\mu_{30}\mu_{50}}$$
$$= \sqrt{(0.000130) \cdot (0.000344)}$$
$$= 0.000211471038 \quad (11)$$

7. Calculate $0.4p_{40.2}$.

**ANSWER:**

By assumption, we have $0.4p_{40.2} = (p_{40})^{0.4} = (0.95)^{0.4}$
8. Calculate $0.2q_{52.4}$, assuming UDD (fractional age assumption)

ANSWER:

We have

\[
0.2q_{52.4} = 1 - 0.2p_{52.4} \\
= 1 - \frac{l_{52.6}}{l_{52.4}} \\
= 1 - \frac{l_{52} - 0.6d_{52}}{l_{52} - 0.4d_{52}} \\
= 1 - \frac{0.4l_{52} + 0.6l_{53}}{0.6l_{52} + 0.4l_{53}} \\
= \frac{l_{52} - l_{53}}{3l_{52} + 2l_{53}} \\
= \frac{89948 - 89089}{859} \\
= \frac{89948 - 89089}{859} \\
= \frac{859}{448022} = 0.00191731656
\]