Basic Information:

\[ V_C(S, t) = e^{-r (T-t)} \mathbb{E} [(S_T - K)_+ | S_t = S] \]
\[ = Se^{-\delta (T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2) \]
\[ V_P(S, t) = e^{-r (T-t)} \mathbb{E} [(K - S_T)_+ | S_t = S] \]
\[ = Ke^{-r(T-t)} N(-d_2) - Se^{-\delta(T-t)} N(-d_1) \]
\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r - \delta + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t} \]
\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz, \quad N(0.15) = 0.559618, \quad N(0.75) = 0.67449. \]

1. Consider the continuous Black Scholes environment with \( \delta = 0 \). What happens to the initial value of a European Put option when \( T \to \infty \)? (10 points.)

**Answer:**

By Put-Call Parity, we know that

\[ V_C(S, t) - V_P(S, t) = S - Ke^{-r(T-t)}. \]

As \( T \to \infty \), we see that the Black-Scholes value of a Call option becomes indistinguishable from the value of the stock:

\[ \lim_{T \to \infty} V_C(S, t) = \lim_{T \to \infty} SN(d_1) - Ke^{-r(T-t)} N(d_2) \]
\[ = S \cdot N(\infty) - K \cdot 0 = S. \]

It follows that

\[ V_P(S, t) = V_C(S, t) - S + Ke^{-r(T-t)} \to S - S + 0 = 0. \]
2. You are considering the purchase of a 3-month, 41.5-strike European call option on a non-dividend-paying stock. You are given:

- The Black-Scholes framework holds.
- The stock is currently selling for 40.
- The stock’s volatility is 30%.
- The current call option delta is 0.5.

Determine

- the risk free rate \( r \) (10 points.)
- the current price of the option (15 points.)

**Answer:**

By our Black-Scholes formula

\[
V^C(40,0) = 40N(d_1) - 41.5e^{-r(\frac{3}{12})}N(d_2)
\]

\[
= 40\Delta - 41.5e^{-r(\frac{3}{12})}N(d_2).
\]

Since our current \( \Delta = 0.5 \), it tells us it is as likely that the option is exercised as it is not. Hence, \( d_1 = 0 \) and so

\[
d_1 = 0 = \frac{\ln\left(\frac{40}{41.5}\right) + (r - 0 + \frac{1}{2}(0.3)^2)(0.25)}{0.3\sqrt{0.25}}
\]

\[
\Rightarrow r = 0.1023
\]

\[
d_2 = 0 - \sigma \sqrt{T} = -0.3\sqrt{0.25} = -0.15
\]

\[
\Rightarrow V^C(40,0) = 40 \left(\frac{1}{2}\right) - 41.5e^{-(0.1023)(\frac{3}{12})}N(-0.15)
\]

\[
= 20 - 40.453N(-0.15)
\]

\[
= 20 - 40.453(1 - N(0.15))
\]

\[
= 40.453N(0.15) - 20.453
\]

\[
= 2.185.
\]
3. Assume the Black-Scholes framework. Consider a nondividend-paying stock, and a European call option and a European put option on the stock.

- The current stock price, call price, and put price are 45.00, 6, and 3, respectively.
- Investor A purchases one calls and one put. Investor B purchases one call and writes (sells) one put.
- The current delta of Investor A for her portfolio is 0.5.

Calculate

- The current delta of Investor B for her portfolio. (10 points)
- \( \Delta^C(S,0), \Delta^P(S,0) \). (15 points)
- Investor B’ Portfolio elasticity. (15 points)

**Hint** For a portfolio with value \( V_{\text{Portfolio}}(S,t) \), the elasticity is

\[
\Omega_{\text{Portfolio}} = \frac{S \cdot \Delta_{\text{Portfolio}}}{V_{\text{Portfolio}}}. \tag{7}
\]

**Answer:**

Recall that or a portfolio of \( M \) options, each with total \( \gamma_i \) and total portfolio value \( V_{\text{Portfolio}} = \sum_{i=1}^{M} \gamma_i V_i \), then

\[
\Delta_{\text{Portfolio}} = \sum_{i=1}^{M} \gamma_i \Delta^i = \sum_{i=1}^{M} \gamma_i \frac{\partial V_i}{\partial S}
\]

\[
\Omega_{\text{Portfolio}} = S \times \frac{\sum_{i=1}^{M} \gamma_i \frac{\partial V_i}{\partial S}}{\sum_{i=1}^{M} \gamma_i V_i}. \tag{8}
\]

and so, by put-call parity for Investor B, we obtain

\[
0.5 = \Delta^A = \Delta^C + \Delta^P
\]

\[
1.0 = \Delta^B = \Delta^C - \Delta^P. \tag{9}
\]

Completing the computation returns
\[ \Delta P = -0.25 \]
\[ \Delta C = 0.75 \]
\[ \Omega^B = \frac{S \Delta^B}{V_B} = \frac{45 \times 1}{6 + 3} = 5. \] (10)
4. For the portfolio of Investor B defined in Question 3, assume that

- both the put and call options were bought sold at the money \( K = 45, \)
- the options were bot written for one year,
- the risk free rate \( r = 0.05, \) and
- the standard deviation of the stock, \( \sigma_{stock} \) is observed to be less that 20%.

With all of these assumptions, calculate the standard deviation of the portfolio of Investor B, \( \sigma_B. \)

(25 points)

**Answer:** Recall the theorem that states \( \sigma_B = \sigma_{stock} \times | \Omega^B |. \)

It follows that \( \sigma_B = 5 \times \sigma, \) where we set \( \sigma = \sigma_{stock} \) and since \( \Delta^C = 0.75 \)

\[
0.67449 = d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r - \delta + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}} \Rightarrow \sigma \in \{0.078724, 1.2703\}
\]

We choose the lower value due to the apriori upper bound on observed stock volatility. It follows that \( \sigma_B = 5 \times 0.078724, \) and so \( \sigma_B = 0.39362, \) or 39.4%. 

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