Important Rules. Read Carefully.

- You must write up your own work. You may be called to my office to explain your answer if I feel that you didn’t answer a question yourself. Be prepared to explain your answer.

- You must submit this exam to me in class by Monday November 9 at 12:40 p.m.

- Your answers must be in this booklet, and must be clearly handwritten. No substitutions accepted.
1. Assume the Black-Scholes framework. The continuously compounded risk-free interest rate is $r$ is unknown, but for a non-dividend paying stock $S$ we know:

- The current stock price $S_0 = 10$
- The stock’s volatility is 10%.
- The price of a 6-month European gap call option on $S$, with a strike price of $K_1 = 10$ and a payment trigger of $K_2 = 9.90$, is 1.
- The price of a 6-month European gap put option on $S$, with a strike price of $K_1 = 10$ and a payment trigger of $K_2 = 9.90$, is 0.50.

The definition of the payoffs is then

$$G_{\text{GapCall}}(S) = (S - K_1)1_{\{S>K_2\}}$$
$$G_{\text{GapPut}}(S) = (K_1 - S)1_{\{S<K_2\}}$$

(1)

Calculate $r$. (10 pts.)

**Answer:**

It follows that

$$C - P = 1 - 0.5 = e^{-rT}\mathbb{E}_0[G_{\text{GapCall}}(S_T) - G_{\text{GapPut}}(S_T)]$$

$$= e^{-rT}\mathbb{E}_0[S_T] - K_1e^{-rT}$$

$$= e^{-rT}S_0e^{rT} - K_1e^{-rT}$$

$$= e^{-rT}S_0e^{rT} - S_0e^{-rT}$$

$$= S_0 \cdot (1 - e^{-rT})$$

$$\Rightarrow e^{-rT} = \frac{S_0 - (C - P)}{S_0}$$

$$\Rightarrow C - P = \frac{10 - 0.5}{10} = 0.95$$

$$\Rightarrow r = -\frac{1}{T} \ln \left( \frac{S_0 - (C - P)}{S_0} \right)$$

$$= -2 \ln (0.95) = 0.102587.$$
2. Assume the Black-Scholes framework.

- An insurance company sells single premium deferred annuity contracts with payment linked to a non-dividend paying stock index.
- The price today of one share of asset $S$ is $S_0 = 100$.
- The contracts offer a minimum guarantee return rate of $100 \cdot \alpha\%$.
- At time 0, a single premium of amount $\pi$ is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company.
- At the contract maturity date, $T = 1$, the insurance company will pay the policyholder

$$G(\pi, S_1) = \pi \times (1 - y) \times \max \left\{ \frac{S_1}{100}, 1 + \alpha \right\}$$

(3)

- The contract is **self-funded**, meaning $\pi$ is exchanged at time 0 in return for the variable payout $G(\pi, S_1)$ at time 1 with no other payments in or out between times 0 and 1.
- $V_0^{Put}(100, 0) = 10$ is the price of a one-year European put option, with strike price of $K = 100 \cdot (1 + \alpha)$, on the stock index.

Calculate the deduction $y$ the company must withdraw so the contract is self-funding under the risk-neutral measure. (20 pts.)

**Answer:**

We tacitly understand that we are using the **EPP** under the risk-neutral pricing measure $\tilde{P}$.

It follows that

$$\pi = e^{-r} \cdot \tilde{E}_0[G(\pi, S_1)]$$

$$= e^{-r} \cdot \frac{\pi \cdot (1 - y)}{100} \cdot \tilde{E}_0[\max \{S_1, 100 \cdot (1 + \alpha)\}]$$

$$= \frac{\pi \cdot (1 - y)}{100} \cdot \tilde{E}_0[S_1 + \max \{0, 100 \cdot (1 + \alpha) - S_1\}]$$

$$= \frac{\pi \cdot (1 - y)}{100} \cdot \left( S_0 + e^{-r} \cdot \tilde{E}_0[\max \{0, 100 \cdot (1 + \alpha) - S_1\}] \right)$$

(4)

$$\Rightarrow y = \frac{V_0^{Put}(100, 0)}{100 + V_0^{Put}(100, 0)} = \frac{10}{110} = 0.091.$$
3. Assume the Black-Scholes framework. At time $t = 0$ you write a $T$-year at-the-money European digital option, where the value is

$$V(S, t) = Ke^{-r(T-t)}N(d_2). \quad (5)$$

Assume that at $t = 0$, you are told $d_2 = 0$.

Calculate:

(a) the initial number $\Delta_0$ of shares of the stock for your hedging program. (10 pts.)

(b) $\sigma_{\text{option}}$, the standard deviation of the option using this $\Delta_0$. (10 pts.)

in terms of the risk free rate $r$ and the term $T$ of the option.

Explain:

(a) Do we also need to know $\sigma$ for the underlying stock? (10 pts.)

(b) What happens to your $\Delta_0$ as $T \to \infty$? (10 pts.)

**Answer:**

Note that since $d_2 = 0$ for our at-the-money option, we know that $r = \frac{1}{2}\sigma^2$. By definition, we can calculate the initial Delta via

$$\Delta(S, t) = \frac{\partial V}{\partial S} = 100e^{-r(T-t)}N'(d_2) \frac{\partial d_2}{\partial S} = \frac{Ke^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}}. \quad (6)$$

\[\Delta_0 := \Delta(K, 0) = \frac{Ke^{-rT}N'(0)}{\sigma K \sqrt{T}} = \frac{e^{-rT}}{\sigma \sqrt{2\pi T}} = \frac{e^{-rT}}{2\sqrt{\pi rT}} \to 0 \text{ as } T \to \infty.\]

Recall the theorem that states $\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega|$. It follows that

$$\sigma_V = \left| \frac{S\Delta}{V} \right| \times \sigma = \frac{K \times \frac{e^{-rT}}{\sigma \sqrt{2\pi T}}}{Ke^{-rT}N'(0)} \times \sigma = \sqrt{\frac{2}{\pi T}}. \quad (7)$$
4. Assuming a continuous Black Scholes framework, a market maker prices a European (call) option to be

\[ V(S, t) = e^{-r(T-t)} \sqrt{S} \]  

(8)

At time \( t \), she observes that \( S_t = 10 \). She presents to her boss that over a time difference of \( dt = \frac{1}{365} \), under a risk-neutral measure \( \tilde{\mathbb{P}} \), the underlying asset changes by the amount

\[
\frac{dS_t}{S_t} = r dt + \sigma Z \sqrt{dt} \\
Z \sim N(0, 1). 
\] 

(9)

- What is the elasticity \( \Omega \)? (10 pts.)
- What is the ratio \( \frac{\tilde{E}[dP_t | S_t = 10]}{V(t)} \) of average incremental Market Maker’s profit-to-Option Value in terms of \( r, \sigma, t, \) and \( T \)? (10 pts.)
- For what value \( r \) is the average incremental Market Maker’s profit \( \tilde{E}[dP_t | S_t = 10] = 0 \)? (10 pts.)

Assume a fully continuous setting, and ignore any values \((dt)^{\gamma}\) for any real number \( \gamma > 1 \).

**Answer:**

Taking the appropriate derivatives,

\[
\begin{align*}
\Delta &= \frac{\partial V}{\partial S} = \frac{V}{2S} \\
\Gamma &= \frac{\partial \Delta}{\partial S} = -\frac{V}{4S^2} \\
\Theta &= \frac{\partial V}{\partial t} = rV \\
\Omega &= \frac{S\Delta}{V} = \frac{S \times \frac{V}{2S}}{V} = \frac{1}{2}.
\end{align*}
\] 

(10)

She computes, for her profit \( P_t \) at time \( t \),
\[ dP_t = d\left( \Delta (S_t, t) \cdot S_t - V(S_t, t) \right) \approx - \left( \Theta dt + r(\Delta S_t - V(S_t, t))dt + \frac{1}{2}\Gamma(dS_t)^2 \right) \]
\[ = - \left( \Theta dt + rV(\Omega - 1)dt + \frac{1}{2}\Gamma(dS_t)^2 \right) \]
\[ \text{(11)} \]

It follows that

\[ \tilde{E}_t[dP_t] = - \left( \Theta dt + rV(\Omega - 1)dt + \frac{1}{2}\Gamma \times \tilde{E}_t[(dS_t)^2] \right) \]
\[ = - \left( \frac{1}{2}rV dt - \frac{1}{2}V \sigma S^2 \tilde{E}[Z^2] \right) \]
\[ = - \frac{1}{2} \left( r - \frac{\sigma^2}{4} \right) V dt \]
\[ \Rightarrow \frac{\tilde{E}[dP_t \mid S_t = 10]}{V(10, t)} = - \frac{1}{2} \left( r - \frac{\sigma^2}{4} \right) dt = - \frac{1}{2} \left( r - \frac{\sigma^2}{4} \right) \frac{1}{365} \]
\[ \text{(12)} \]

From the above calculation, it follows that there is no average profit under the risk-neutral measure if \( r = \frac{\sigma^2}{4} \).