Important Rules. Read Carefully.

• You must write up your own work. You may be called to my office to explain your answer if I feel that you didn’t answer a question yourself. Be prepared to explain your answer.

• You must submit this exam to me in class by Monday November 9 at 12:40 p.m.

• Your answers must be in this stapled booklet, and must be clearly handwritten. No substitutions accepted.
1. Assume the Black-Scholes framework. The continuously compounded risk-free interest rate is $r$ is unknown, but for a non-dividend paying stock $S$ we know:

- The current stock price $S_0 = 10$
- The stock’s volatility is 10%.
- The price of a 6-month European gap call option on $S$, with a strike price of $K_1 = 10$ and a payment trigger of $K_2 = 9.90$, is 1.
- The price of a 6-month European gap put option on $S$, with a strike price of $K_1 = 10$ and a payment trigger of $K_2 = 9.90$, is 0.50.

The definition of the payoffs is then

$$
G_{\text{GapCall}}(S) = (S - K_1)1_{\{S > K_2\}}
$$

$$
G_{\text{GapPut}}(S) = (K_1 - S)1_{\{S < K_2\}}
$$

(1)

Calculate $r$. (10 pts.)
2. Assume the Black-Scholes framework.

- An insurance company sells single premium deferred annuity contracts contracts with payment linked to a non-dividend paying stock index.
- The price today of one share of asset \( S \) is \( S_0 = 100 \).
- The contracts offer a minimum guarantee return rate of \( 100 \cdot \alpha \% \).
- At time 0, a single premium of amount \( \pi \) is paid by the policyholder, and \( \pi \times y\% \) is deducted by the insurance company.
- At the contract maturity date, \( T = 1 \), the insurance company will pay the policyholder

\[
G(\pi, S_1) = \pi \times (1 - y) \times \max \left\{ \frac{S_1}{100}, 1 + \alpha \right\}
\]  

(2)

- The contract is **self-funded**, meaning \( \pi \) is exchanged at time 0 in return for the variable payout \( G(\pi, S_1) \) at time 1 with no other payments in or out between times 0 and 1.

- \( V^P_0(100, 0) = 10 \) is the price of a one-year European put option, with strike price of \( K = 100 \cdot (1 + \alpha) \), on the stock index.

Calculate the deduction \( y \) the company must withdraw so the contract is self-funding under the risk-neutral measure. (20 pts.)
3. Assume the Black-Scholes framework. At time $t = 0$ you write a $T$-year at-the-money European digital option, where the value is

$$V(S, t) = K e^{-r(T-t)} N(d_2).$$

Assume that at $t = 0$, you are told $d_2 = 0$.

Calculate:

(a) the initial number $\Delta_0$ of shares of the stock for your hedging program. (10 pts.)

(b) $\sigma_{\text{option}}$, the standard deviation of the option using this $\Delta_0$. (10 pts.)

in terms of the risk free rate $r$ and the term $T$ of the option.

Explain:

(a) Do we also need to know $\sigma$ for the underlying stock? (10 pts.)

(b) What happens to your $\Delta_0$ as $T \to \infty$? (10 pts.)
4. Assuming a continuous Black Scholes framework, a market maker prices a European (call) option to be

\[ V(S, t) = e^{-rt(T-t)} \sqrt{S} \]  

At time \( t \), she observes that \( S_t = 10 \). She presents to her boss that over a time difference of \( dt = \frac{1}{365} \), under a risk-neutral measure \( \tilde{P} \), the underlying asset changes by the amount

\[ \frac{dS_t}{S_t} = rdt + \sigma Z \sqrt{dt} \quad Z \sim N(0, 1). \]  

- What is the elasticity \( \Omega \)? (10 pts.)
- What is the ratio \( \frac{\tilde{E}[dP_t | S_t = 10]}{V(S_t, t)} \) of average incremental Market Maker’s profit-to-Option Value in terms of \( r, \sigma, t, \) and \( T \)? (10 pts.)
- For what value \( r \) is the average incremental Market Maker’s profit \( \tilde{E}[dP_t | S_t = 10] = 0 \)? (10 pts.)

Assume a fully continuous setting, and ignore any values \( (dt)^\gamma \) for any real number \( \gamma > 1 \).