1. Consider a one-period, trinomial financial model where $S_0 = 400$, $r = 0.05$ and

$$S_1(\omega_3) = 500$$
$$S_1(\omega_2) = 400$$
$$S_1(\omega_1) = 300.$$ \hspace{1cm} (1)

Now, consider you are also given, for

$$V_{1^{\text{digital}}} = \begin{cases}
1 & : S_1(\omega) > 401 \\
0 & : S_1(\omega) \leq 401,
\end{cases}$$

the value

$$V_0^{\text{digital}} = \frac{1}{1+r} \tilde{\mathbb{E}}_0[V_1^{\text{digital}}] = 0.25.$$ \hspace{1cm} (2)

Is this model arbitrage free? Is this model complete? Is there a risk neutral measure that you can compute? If so, what is it (them)? If possible, compute the arbitrage-free price of

- A standard put with strike $K = 420$. (15 pts.)
- A standard call with strike $K = 420$. (10 pts.)

**Solution:** Yes, this model is complete, as there are three states and three equations to solve:

- $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1$ \textit{[due to the fact we have three states.]}
- $\frac{1}{1.05} (500\tilde{p}_1 + 400\tilde{p}_2 + 300\tilde{p}_3) = 400$ \textit{[Risk Neutral price of Stock.]}
- $\frac{1}{1.05} (1\tilde{p}_1 + 0\tilde{p}_2 + 0\tilde{p}_3) = 0.25$ \textit{[Risk Neutral Price of Digital Option with K = 401.]}

Solving this system of equations by finding the rref of the matrix \( M \):

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 \\
500 & 400 & 300 & 420 \\
1 & 0 & 0 & 0.25(1.05)
\end{bmatrix}
\]

results in

\[
\text{rref}(M) = \begin{bmatrix}
1 & 0 & 0 & 0.2625 \\
0 & 1 & 0 & 0.675 \\
0 & 0 & 1 & 0.0625
\end{bmatrix}
\]

It follows that \((\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = (0.2625, 0.675, 0.0625)\), and so

\[
V_0^C = \frac{1}{1.05} \tilde{E}[(S_1 - 420)_+ | S_0 = 400] \\
= 0.2625 \times (500 - 420) \\
= 20
\]

\[
V_0^P = \frac{1}{1.05} \tilde{E}[(420 - S_1)_+ | S_0 = 400] \\
= 0.675 \times (420 - 400) + 0.0625 \times (420 - 300) \\
= 20.
\]

Could also have used put-call parity.

2. Consider a financial model where there are two currencies, foreign and domestic:

- \( S_A^B = 4 \) is the spot exchange rate - one unit of B is worth \( S_A^B \) of A today (time 0).
- \( r^A = 0.1 \) is the domestic borrow/lend rate.
- \( r^B = 0 \) is the foreign borrow/lend rate, as you are not allowed to invest in the foreign banking system.
- The spot rate at time 1 can be modeled as either binomial or trinomial - your choice!
- Compute the forward exchange rate \( F_A^B \). This is the value of one unit of B delivered at time 1 in return for \( F_A^B \), where \( F_A^B \) is effectively locked in at time 0. (15 pts.)
SOLUTION:

- **Method of Replication**
  The process to price via replication is
  - At time 1, we deliver 1 unit of B in exchange for $F^B_A$ units of domestic currency A.
  - This is a forward contract - we pay nothing up front to achieve this.
  - Initially borrow some amount foreign currency B, in foreign market to grow to one unit of B at time 1. This is achieved by the initial amount $\frac{S^B_A}{1+r_B}$ (valued in domestic currency.)
  - Invest the amount $\frac{F^B_A}{1+r_A}$ in domestic market (valued in domestic currency.)
  - This results in the initial value
    \[
    V(0) = \frac{F^B_A}{1+r_A} - \frac{S^B_A}{1+r_B}
    \]  
    (6)
  
  Since the initial value is 0, this means
  \[
  F^B_A = S^B_A \frac{1+r_A}{1+r_B} = S^B_A (1+r_A) = 4.4.
  \]  
  (7)

- **Method of Forward Contracts**
  - To compute this using a standard forward contract, note that in our economy we would expect for an asset that pays no dividends,
    \[
    F^B_A = S^B_A (1+r_A) = 4.4.
    \]  
    (8)

3. Assume a discrete model for the evolution of the stock, except now we do not know if the model is binomial, trinomial, or even $n$ states $\{\omega_1, \ldots, \omega_n\}$ at time 1. However, we do know that the discount rate is $r = 0.05$ and that there there is a market for three options $V^A, V^C, V^P$ defined by the following payoffs at time 1:

\[
V^A(S_1, 1) = |S_1 - K|
\]
\[
V^C(S_1, 1) = (S_1 - K)_+
\]
\[
V^P(S_1, 1) = (K - S_1)_+
\]  
(9)
where \( K \) is a strike chosen by the option purchaser. We will assume that a unique risk-neutral measure \( \tilde{P} \) exists in this world.

Recall under the risk-neutral measure \( \tilde{P} \), the price of an option with general payoff \( V(S_1, 1) = G(S_1) \)

is

\[
V(S, 0) = \frac{1}{1 + r} \tilde{E}[G(S_1) \mid S_0 = S]
\]  

(10)

An analysis of past data shows that the market for options \( A \) and \( C \) have their prices as approximately

\[
\begin{align*}
V^A(S_0, 0) &= 100 \\
V^C(S_0, 0) &= 50.
\end{align*}
\]  

(11)

when \( K = 80 \). Is there enough data here to find the initial stock price \( S_0 \)? If so, what is \( S_0 \)? (10 pts.)

**Solution:** Yes, there is enough information even if we don’t know the exact stochastic process for \( S_t \) over times \( 0 \leq t \leq 1 \). In fact, we know that for a put option payoff \( V^P(S_1, 1) \) at time 1, we have the relationship

\[
V^A(S_1, 1) = V^C(S_1, 1) + V^P(S_1, 1).
\]  

(12)

It follows that under the risk-neutral pricing measure, the value of

\[
V^P(S_0) = V^A(S_0, 0) - V^C(S_0, 0) = 100 - 50 = 50
\]

\[
\Rightarrow \frac{F - K}{1 + r} = 50 - 50 = 0
\]

\[
\Rightarrow S_0 = \frac{F}{1 + r} = \frac{K}{1 + r} = \frac{80}{1.05} = 76.19.
\]  

(13)