1. Consider a one-period, trinomial financial model where \( S_0 = 200 \), \( r = 0.06 \) and

\[
\begin{align*}
S_1(\omega_3) &= 220 \\
S_1(\omega_2) &= 212 \\
S_1(\omega_1) &= 200
\end{align*}
\]

and one can buy a Call option with strike \( K = 212 \) for \( V_0^C = 2 \) today.

Is this model arbitrage free? Is this model complete? Is there a risk neutral measure that you can compute? If so, what is it (them)? If possible, compute the arbitrage-free price of

- A standard put with strike \( K = 205 \). (15 pts.)
- A standard call with strike \( K = 215 \). (10 pts.)

**Solution:** Yes, this model is complete, as there are three states and three equations to solve:

- \( \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1 \) [due to the fact we have three states]
- \( 220\tilde{p}_1 + 212\tilde{p}_2 + 200\tilde{p}_3 = 200(1.06) \) [Risk Neutral price of Stock]
- \( 8\tilde{p}_1 + 0\tilde{p}_2 + 0\tilde{p}_3 = 2(1.06) \) [Risk Neutral Call Price with \( K = 212 \)]

Solving this system of equations yields

\[
(p_1, p_2, p_3) = (0.265, 0.558, 0.176).
\]

Using these risk-neutral probabilities, we compute

\[
\begin{align*}
V_0^P &= \frac{1}{1.06} \tilde{E}[\max\{205 - S_1, 0\}] = 0.83 \\
V_0^C &= \frac{1}{1.06} \tilde{E}[\max\{S_1 - 215, 0\}] = 1.25.
\end{align*}
\]
2. Consider a financial model consisting of a stock and bank account, with a constant monthly interest rate of 1%. Based on this model, we wish to price a financial derivative that, at the end of one year, pays the monthly average over the term. Symbolically, if we consider months $i = 1, 2, \ldots, 11, 12$, the payoff at time $i = 12$ is

$$V_{12} := \frac{1}{12} \sum_{i=1}^{12} S_i$$

If $S_0 = 100$, what is the arbitrage free price $V_0$ of this security? (15 pts.)

**SOLUTION:**

- **Method of Replication** To replicate the final payout, we simply have to buy some amount of stock and sell it off, piece by piece, to raise the capital needed to deliver the payout required by the contract. If we sell an amount $\alpha(i)$ of stock at time $i$, then the value of that sale is

$$\alpha(i)S_i$$

Of course, by time 12, this will have grown to value

$$\alpha(i)S_i(1.01)^{12-i} = \frac{1}{12} S_i$$

It follows that we need to buy

$$\alpha(i) := \frac{1}{12 \cdot (1.01)^{12-i}}$$

shares at time 0 to sell at time $i$. This means that in total, we need

$$\frac{1}{12} \sum_{i=1}^{12} \frac{1}{1.01^{12-i}} = \frac{1}{12 \cdot 1.01^{12}} \cdot \frac{1.01^{13} - 1.01}{0.01} = 0.9473$$

shares at time 0. It follows that the price of this security at time 0 is 94.73.
• Method of Multiple Forward Contracts

• To compute this using a risk neutral measure, notice that since

\[ \tilde{E}_0 [S_i] = S_0(1.01)^i, \]

\[
V(S_0, 0) = \frac{1}{(1.01)^{12}} \tilde{E}_0 \left[ \frac{1}{12} \sum_{i=1}^{12} S_i \right]
\]

\[
= \frac{1}{(1.01)^{12}} \frac{1}{12} \sum_{i=1}^{12} \tilde{E}_0 [S_i]
\]

\[
= \frac{1}{(1.01)^{12}} \frac{1}{12} \sum_{i=1}^{12} S_0(1.01)^i
\]

\[
= \frac{S_0}{12} \sum_{i=1}^{12} (1.01)^{i-12} = 0.9473S_0 = 94.73.
\]

(9)
3. Assume a discrete model for the evolution of the stock, except now we do not know if the model is binomial, trinomial, or even $n$ states $\{\omega_1, ..., \omega_n\}$ at time 1. However, we do know that the discount rate is $r = 0.25$ and that there is a market for the following two options $V^A$ and $V^B$ defined by the following payoff conditions:

$$V^A(S_1, 1) = \frac{1}{2}S_1 - \frac{1}{2}K + \frac{e^{S_1} - 1}{2} \quad (10)$$

$$V^B(S_1, 1) = \frac{1}{2}S_1 + \frac{1}{2}K + \frac{1 - e^{S_1}}{2} \quad (11)$$

where $K$ is a strike chosen by the option purchaser. We will assume that a risk-neutral measure $\tilde{P}$ exists in this world.

Recall under the risk-neutral measure $\tilde{P}$, the price of an option with general payoff $V(S_1, 1) = G(S_1)$ is

$$V(S, 0) = \frac{1}{1 + r} \tilde{E}[G(S_1) \mid S_0 = S] \quad (12)$$

An analysis of past data shows that the market for options $A$ and $B$ have their prices as approximately

$$V^A(S, 0) = 1.1S + 0.05S^2 - \frac{1}{2}K$$

$$V^B(S, 0) = -0.05S^2 + \frac{1}{2}K \quad (13)$$

where $K$ is again left to the buyer to determine. Is there enough data here to find the price $F$ of a Forward contract on a unit of stock $S_1$ at time $T = 1$? If so, what is $F$ in terms of the initial stock price $S$ at time 0? (10 pts.)
Solution: Yes, there is enough information even if we don’t know the exact stochastic process for $S_t$ over times $0 \leq t \leq 1$. In fact, under the risk-neutral pricing measure,

\[
F = \tilde{E} \left[ S_1 \mid S_0 = S \right] \\
= \tilde{E} \left[ \frac{1}{2} S_1 - \frac{1}{2} K + \frac{e^{S_1} - 1}{2} \mid S_0 = S \right] + \tilde{E} \left[ \frac{1}{2} S_1 + \frac{1}{2} K + \frac{1 - e^{S_1}}{2} \mid S_0 = S \right] \\
= (1 + r) \left( V^A(S, 0) + V^B(S, 0) \right) \\
= (1.25) (1.1S) \\
= 1.375S
\]

(14)