Credit Spreads with Stochastic Recovery : Connections with PDE's and Probability

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Outline

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Transformation of Stochastic Processes

In Stochastic Calculus, we are often interested in transforming collection of stochastic processes into other stochastic processes. For example, for a pair of processes (A_t, L_t) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a measurable function h, define

$$Y_t = h(P_t, L_t, t). \tag{1}$$

- Applications in modeling in science and economics.
- Ito Calculus.
- Used in linking PDEs and Probability

Connection with Bond Pricing and Credit Risk

For zero-coupon bonds with

- **risk-free rate** r
- **Maturity** T
- **Redemption Value** $\bar{B}$
- **Credit Spread** $Y_{t,T}$ from $t \to T$,

it follows that $B_{t,T}$, the **bond price at time** t, can be modeled in continuous time as

$$B_{t,T} = \bar{B} e^{-(r+Y_{t,T})(T-t)}. \tag{2}$$

If *no default is possible,* then $Y_{t,T} \equiv 0$ for all (t, T).

How Do We Model Credit Spread Under Default?

One approach is to use a structural approach, where the assets of a company at time t, A_t, are used to determine the credit spread and associated metrics for bond prices:

$$
\begin{aligned}
Y_{t,T} &= g(A_t, t) \\
&= \frac{1}{T-t} \ln\left(\frac{\bar{B}}{B_t}\right) - r \\
&= \frac{1}{T-t} \ln\left(\frac{1}{1 - \tilde{\mathbb{P}}_t[\textbf{Default}]\tilde{\mathbb{E}}_t[\textbf{Loss} \mid \textbf{Default}]}\right)
\end{aligned} \tag{3}
$$

- A very important question is how $\tilde{\mathbb{P}}_t[\textbf{Default}]$ and $\tilde{\mathbb{E}}_t[\textbf{Loss} \mid \textbf{Default}]$ are correlated .
- Empirical evidence shows that they are.
- Apparent in a recession, for example.

Merton Model

Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the existence of a risk-neutral measure where

$$
\begin{aligned}
B_{t,T} = f(A_t, t) &= e^{-r(T-t)}\tilde{\mathbb{E}}_t\Big[\min\left\{\bar{B}, A_T\right\}\Big] \\
S_t = A_t - B_{t,T} &= e^{-r(T-t)}\tilde{\mathbb{E}}_t\Big[(A_T - \bar{B})_+\Big] \\
dA_t &= \mu_A A_t dt + \sigma_A A_t dW_t^A \\
&= rA_t dt + \sigma_A A_t d\tilde{W}_t^A \\
\textbf{Default} &= \left\{A_T \geq \bar{B}\right\}^c \\
\tilde{\mathbb{P}}_t[\textbf{Default}] &= \tilde{\mathbb{P}}_t[\left\{A_T \geq \bar{B}\right\}^c] \\
\tilde{\mathbb{E}}_t[\textbf{Loss} \mid \textbf{Default}] &= \tilde{\mathbb{E}}_t\Big[\frac{\bar{B} - A_T}{\bar{B}} \mid \left\{A_T \geq \bar{B}\right\}^c\Big].
\end{aligned} \tag{4}
$$

Merton Model Bond Price

Consequently, we have a differental operator

$$\tilde{L}_A := \partial_t + rA\partial_A + \frac{1}{2}\sigma_A^2\partial_{AA} - r \quad (5)$$

and the final-value problem for $B = f(A, t)$

$$\begin{aligned} \tilde{L}_A[f] &= 0 \\ f(A, T) &= \min\left\{\bar{B}, A\right\}. \end{aligned} \quad (6)$$

Merton Model Bond Price, PD, and LGD

$$
\begin{aligned}
B_{t,T}^{\text{Merton}} &= \bar{B}e^{-r(T-t)}\mathcal{N}(d_0) + A_t\mathcal{N}(-d_1) \\
PD_{\text{Merton}} &= \tilde{\mathbb{P}}_t[A_T < \bar{B}] = \mathcal{N}(-d_0) \\
LGD_{\text{Merton}} &= \frac{1}{\bar{B}}\tilde{\mathbb{E}}_t[\bar{B} - A_T | A_T < \bar{B}] = 1 - e^{r(T-t)}\frac{A_t}{\bar{B}}\frac{\mathcal{N}(-d_1)}{\mathcal{N}(-d_0)} \\
Y_{\text{Merton}}(t,T) &= -\frac{1}{T-t}\ln\left[\mathcal{N}(d_0) + \frac{A_t}{\bar{B}}e^{r(T-t)}\mathcal{N}(-d_1)\right] \qquad (7) \\
d_1 &= \frac{\ln(A_t/\bar{B}) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} = d_+ \\
d_0 &= \frac{\ln(A_t/\bar{B}) + (r - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A\sqrt{T-t}} = d_-.
\end{aligned}
$$

Black-Cox Model Setup

- Same asset dynamics as in Merton model:
- Default can happen at times other than maturity, in particular when the assets fall below a prescribed default point DP:

$$\begin{aligned}\textbf{Default} &= \left\{A_T \geq \bar{B}, \tau_{DP} > T\right\}^c \\ \tilde{\mathbb{P}}_t[\textbf{Default}] &= \tilde{\mathbb{P}}_t[\left\{A_T \geq \bar{B}, \tau_{DP} > T\right\}^c] \\ \tilde{\mathbb{E}}_t[\textbf{Loss} \mid \textbf{Default}] &= \tilde{\mathbb{E}}_t\left[\frac{\bar{B} - A_T}{\bar{B}} \mid \left\{A_T \geq \bar{B}, \tau_{DP} > T\right\}^c\right].\end{aligned} \tag{8}$$

and the final-boundary-value problem for $B = f(A, t)$

$$\begin{aligned}\tilde{L}_A[f] &= 0 \\ f(A, T) &= \min\left\{\bar{B}, A\right\} \\ f(DP, t) &= DP.\end{aligned} \tag{9}$$

Bond Price under Correlated A-R Model

Define the quantities:

- A_t, the asset value at time $t > 0$.
- R_t, the recovery amount at time $t > 0$. We model recovery as a "shadow" asset, and thus follows the same dynamics as the asset.

$$\begin{aligned} \frac{dA_t}{A_t} &= r_f dt + \sigma_A d\tilde{W}_t^A \\ \frac{dR_t}{R_t} &= r_f dt + \sigma_R d\tilde{W}_t^R \\ \tilde{\mathbb{E}}[dW_t^A dW_t^R] &= \rho_{A,R} dt. \end{aligned} \tag{10}$$

The new two factor differential operator is

$$\tilde{L}_{A,R} := \tilde{L}_A + rR\partial_R + \frac{1}{2}\sigma_{RR}^2 \partial_{RR} + \rho_{A,R}\partial_{AR} \tag{11}$$

Two Factor Merton Model - Default at Maturity

If we allow for a shadow recovery process R, then we are left with an *exchange option* where the payoff at expiry is

$$G(A_T, R_T) = \bar{B} 1_{\{A_T \geq \bar{B}\}} + R_T 1_{\{A_T < \bar{B}\}}. \tag{12}$$

Our two-factor FVP is

$$\begin{aligned} \tilde{L}_A[B] &= 0 \\ B(A, R, T) &= G(A, R). \end{aligned} \tag{13}$$

Two Factor Merton Model - Default at Maturity

Consequently, our results show that

$$
\begin{aligned}
B_t &= e^{-r_f(T-t)}\tilde{\mathbb{E}}_t\left[\bar{B}1_{\{A_T \geq \bar{B}\}} + R_T 1_{\{A_T < \bar{B}\}}\right] \\
&= \bar{B}e^{-r_f(T-t)} \cdot \tilde{\mathbb{P}}_t[A_T \geq \bar{B}] + \tilde{\mathbb{E}}_t\left[e^{-r_f(T-t)} R_T 1_{\{A_T < \bar{B}\}}\right] \\
&= \bar{B}e^{-r_f(T-t)}\mathcal{N}(d_0) + R_t\mathcal{N}(-d_\gamma) \\
\gamma &:= \rho_{A,R}\frac{\sigma_R}{\sigma_A} \\
d_\gamma &= \frac{\ln(A_t/\bar{B}) + (r_f - \frac{1}{2}\sigma_A^2 + \gamma)(T-t)}{\sigma_A\sqrt{T-t}}.
\end{aligned}
\tag{14}
$$

Note that

- We retain the classical $1d$ Merton price and recovery metrics if
  - $\rho_{A,R} = \gamma = 1$ and
  - $\tilde{\mathbb{P}}_t[A_t = R_t] = 1$.
- γ is the financial elasticity of the recovery R to the "benchmark" asset A. This is also known as the β of R to A in the equities framework.

Two Factor Merton Model - Default at Maturity

Recovery metrics:

$$
\begin{aligned}
\tilde{\mathbb{E}}_t[\textbf{Loss} \mid \textbf{Default}] &= \frac{\tilde{\mathbb{E}}_t[\bar{B} - R_T \mid \{A_T \geq \bar{B}\}^c]}{\bar{B}} \\
&= 1 - e^{r_f(T-t)} \frac{R_t}{\bar{B}} \frac{\mathcal{N}(-d_\gamma)}{\mathcal{N}(-d_0)} \\
\mathbb{P}_t[\textbf{Default}] &= \mathcal{N}(-d_0) \\
Y_{\text{2D Merton}}(t, T) &= \frac{1}{T-t} \ln\left(\frac{1}{1 - \tilde{\mathbb{P}}_t[\textbf{Default}]\tilde{\mathbb{E}}_t[\textbf{Loss} \mid \textbf{Default}]} \right)
\end{aligned}
\tag{15}
$$

Note that only **LGD** has changed in this model from the $1d$ Merton setting. The **PD** remains the same as in the $1d$ Merton setting.

Main Theorem: Two Factor Black-Cox Model: $DP < \bar{B}$

For $B_{t,T}^{\text{2d Black Cox}} = h(A_t, R_t, t)$, where h satisfies

$$\begin{aligned} \tilde{L}_A[h] &= 0 \\ h(A, R, T) &= G(A, R) \\ f(DP, R, t) &= DP, \end{aligned} \tag{16}$$

we can solve in closed form to retain:

$$\begin{aligned} B_{t,T}^{\text{2d Black Cox}} &= \bar{B} e^{-r_f(T-t)} \left[\mathcal{N}(d_0) + \left(\frac{DP}{A_t} \right)^{\delta_0} \mathcal{N}(x_0) \right] \\ &\quad + R_t \left[\mathcal{N}(-d_\gamma) + \left(\frac{DP}{A_t} \right)^{\delta_\gamma} \mathcal{N}(x_\gamma) \right] \\ \delta_\gamma &= \frac{2r}{\sigma_A^2} - 1 + 2\gamma \\ x_\gamma &= \frac{\ln(DP^2/\bar{B}A_t) + (r_f - \frac{1}{2}\sigma_A^2 + \gamma)(T-t)}{\sigma_A \sqrt{T-t}}. \end{aligned} \tag{17}$$

Consistency: Reduction to 1-D Black-Cox Model

As a result of the closed form above for $B_{t,T}$, the consistency of the model follows quickly:

Lemma

$$\lim_{T \to t^+} B_{t,T}^{2d\ Black\ Cox} = \bar{B}$$
$$\lim_{(R_t,\rho_{A,R},\sigma_R) \to (A_t,1,\sigma_A)} B_{t,T}^{2d\ Black\ Cox} = B_{t,T}^{1d\ Black\ Cox} \tag{18}$$

Conclusions and Next Papers?

- Have extended 2d models to include closed form solutions (both Merton and Black-Cox)
- Recovery is unbounded. Historically possible (foreclosure and bank fees.)
- Can include bounded recovery in this framework, work underway.
- Can extend to other products, such as Credit Linked Notes.
- Calibration to market data for both CDS and Bond prices (any takers?)

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