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SFU Actuarial Seminar - June 2014
Outline

1. Review of Some Structural Models
   - Fundamentals of Structural Models
   - Merton Model
   - Black-Cox Model
   - Moody’s KMV Model

2. Stochastic Recovery
   - Complete Recovery Model
   - Partial Recovery Model
   - Correlated Asset-Recovery Model
   - Bond Price - 2d Black-Cox Model
   - CDS - 2d Black-Cox Model
   - Conclusions

Joint work with Nick Costanzino (nick_costanzino@otpp.com, OTPP and U of Toronto - RiskLab)
Single-name default models typically fall into one of three main categories:

- **Structural Models.** Attempts to explain default in terms of fundamental properties, such as the firm's balance sheet and economic conditions (Merton 1974, Black-Cox 1976, Leland 1994 etc).

- **Reduced Form (Intensity) Models.** Directly postulate a model for the instantaneous probability of default via an exogenous process $\lambda_t$ (Jarrow-Turnbull 1995, Duffie-Singleton 1999 etc) via:

  $$P[\tau \in [t, t+dt) | F_t] = \lambda_t dt$$

- **Hybrid Models.** Incorporates features from structural and reduced-form models by postulating that the default intensity is a function of the stock or of firm value (Madan-Unal 2000, Atlan-Leblanc 2005, Carr-Linetsky 2006 etc).
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What Exactly Are Structural Models Supposed To Do?

Price both corporate debt and equity securities (perhaps even equity derivatives and CDS!)

Important for buyers/investors, sellers/firms, and advisors

Estimate default probabilities of firms.

Useful to commercial banks, investment banks, rating agencies etc.

Determining optimal capital structure decisions.

Essential part of economic capital estimation (Moody's Portfolio Manager and RiskFrontier)

Analyzing most corporate decisions that affects cash flows.

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How Do We Mathematically Define Default?

Several possible triggers for default:

- **Zero Net Worth Trigger.** Asset value falls below debt outstanding. But... firms often continue to operate even with negative net worth. Might issue more stock and pay coupon rather than default.

- **Zero Cashflow Trigger.** Cashflows inadequate to cover operating costs. But... zero current net cash flow doesn’t always imply zero equity value. Shares often have positive value in this scenario. Firm could again issue more stock rather than default.
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Simplest Framework for Structural Models

Assume that the firm’s assets $A_t$ is the sum of equity $E_t$ and debt $D_t$.

Assumptions on the assets process must be postulated.

Assumptions on the debt structure must be postulated.

A default mechanism $\tau$ and payout structure $f(A_\tau, \tau)$ must be postulated.

The main idea, pioneered by Merton, is that the firm’s equity $E_t$ is then modeled as a call option on the assets $A_t$ of the firm at expiry $T$:

\[ E_t = V(A_t, t) = \tilde{E}[e^{-r(T-t)}(A_T - \bar{B}) + \mid A_t = A_t] \]

This connects equity $E_t$ with debt value $D_t$. In particular if $D_t$ has a tractable debt structure it provides a methodology to compute default probability, bond price, credit spreads, recovery rates, etc.
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Merton Model Set-up

Assume a probability space \((\Omega, F, P)\).

Underlying asset is modeled as GBM under Physical Measure \(P\):

\[
dA_t = \mu A_t dt + \sigma A_t dW_t
\]

Default is implicitly assumed to coincide with the event \(\{A_T < \bar{B}\}\):

\[
\tau_{Merton} = T \{A_T < \bar{B}\} + \infty \{A_T \geq \bar{B}\}
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This results in a turnover of the company's assets to bondholders if assets are worth less than the total value of bond outstanding.

At maturity the bond value (payoff) is:

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\[
= e^{-r(T-t)} \left[ \int_0^{\bar{B}} A \cdot \tilde{P}_t[A_T \in dA] + \bar{B} \int_{\bar{B}}^{\infty} \tilde{P}_t[A_T \in dA] \right]
\]

\[
= e^{-r(T-t)} \tilde{E}_t[A_T \mid A_T < \bar{B}] \cdot \tilde{P}_t[A_T < \bar{B}] + e^{-r(T-t)} \bar{B} \cdot \tilde{P}_t[A_T \geq \bar{B}]
\]

\[
= \bar{B} e^{-r(T-t)} - e^{-r(T-t)} \tilde{E}_t[(\bar{B} - A_T) \mid A_T < \bar{B}] \cdot \tilde{P}_t[A_T < \bar{B}]
\]

\[
:= \bar{B} e^{-r(T-t)} - \bar{B} e^{-r(T-t)} \tilde{E}_t[\text{Loss} \mid \text{Default}] \cdot \tilde{P}_t[\text{Default}].
\]

Here, **Loss** denotes the loss per unit of face in the event \(\{A_T < \bar{B}\}\).
Merton Model PD and LGD

\[ B_{t,T}^{\text{Merton}} = \bar{B} e^{-r(T-t)} \mathcal{N}(d_2) + A_t \mathcal{N}(-d_1) \]

\[ PD_{\text{Merton}} = \tilde{P}_t [A_T < \bar{B}] = \mathcal{N}(-d_2) \]

\[ LGD_{\text{Merton}} = \frac{1}{\bar{B}} \mathbb{E}_t [\bar{B} - A_T | A_T < \bar{B}] = 1 - e^{r(T-t)} \frac{A_t}{\bar{B}} \frac{\mathcal{N}(-d_1)}{\mathcal{N}(-d_2)} \]  \quad (1)

\[ d_1 = \frac{\ln(A_t/\bar{B}) + (r + \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A \sqrt{T-t}} = d_+ \]

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Real-world probabilities from risk-neutral probabilities via \( \mu \to r \) (Wang Transform:)
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Real-world probabilities from risk-neutral probabilities via \( \mu \rightarrow r \) (Wang Transform:)

\[ PD_{\text{Merton}}^{\mathcal{P}} = \mathcal{N}\left(\mathcal{N}^{-1}(PD_{\text{Merton}}^{\mathcal{Q}}) - \frac{\mu_A - r}{\sigma_A} \sqrt{T}\right) \]
The yield-to-maturity credit spread $Y(t, T)$ is defined as the spread over the risk free rate $r$ which prices the bond:

$$B_t = \bar{B}e^{-r(T-t)} - (r + Y(t, T))(T-t)$$

Solving for $Y$ yields $Y(t, T) = \frac{1}{T-t} \ln\left(\frac{\bar{B}}{B_t}\right) - r$

For the Merton Model:

$$Y_{Merton}(t, T) = -\frac{1}{T-t} \ln\left[N(d_2) + \frac{A_t}{\bar{B}e^{r(T-t)}} N(-d_1)\right]$$

Albert Cohen (MSU)
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Merton Model Parameter Estimation

To compute the desired quantities, model asks for $A$ and $\sigma_A$ as inputs. These are unobservable from the markets. The solution is to imply them from the market.
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Note that these equations above are obtained via

\[ E_t = V(A, t) = \tilde{E} \left[ e^{-r(T-t)} (A_T - \bar{B})_+ \mid A_t = A \right] \]

\[ \frac{\sigma_V}{\sigma_A} = \left| \frac{A \frac{\partial V}{\partial A}}{V(A, t)} \right|. \]
Merton Model Criticisms

The assumption that the firm can only default at debt maturity is empirically violated. Model tends to underestimate default probabilities and credit spreads, especially for short maturities. Indeed

$$T \rightarrow t + 1$$

$$T - t \quad P_t \left[ A_T \leq \bar{B} \right] = 0$$

Does not take into account empirical evidence that recovery is correlated to probability of default.

Does not take into account liquidity issues.
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Black-Cox Model Setup

Same asset dynamics as in Merton model:
\[ dA_t = \mu_A A_t dt + \sigma_A A_t dW_A \]

Default can happen at times other than maturity, in particular when the assets fall below a prescribed default point \( DP \):

\[ \tau_{Black-Cox} = \min \{ \inf \{ t \geq 0 : A_t \leq DP \}, \tau_{Merton} \} \]

In the original Black-Cox paper, \( DP \) was taken to be:
\[ DP = C_e - \gamma (T - t) \]

The payoff to the bondholder is:
\[ \text{payoff}_{Black-Cox} = A_{\tau} 1_{\{\tau \leq T\}} + \bar{B} 1_{\{\tau > T\}} \]
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  \[ DP = Ce^{-\gamma(T-t)} \]

- The payoff to the bondholder is
  \[ \text{payoff}_{\text{Black–Cox}} = A_T 1_{\{\tau \leq T\}} + \bar{B} 1_{\{\tau > T\}} \]
KMV Model Setup

Moody's starts by defining a "distance-to-default" (DD):

$$DD = \ln A_{DP} + (\mu_A - \sigma_A^2/2)(T - t)/\sigma_A \sqrt{T - t}$$

which is just $d_2$ in the Merton model with $\overline{B}$ replaced with $DP$ and $r$ replaced with $\mu_A$.

Default point $DP$ chosen as the short term debt plus half of long-term debt:

$$DP = ST + \frac{1}{2} LT$$

Then calibrate DD to empirical distribution $\Psi$ of realized defaults from proprietary database:

$$PD_Moody' = \Psi(-DD)$$

Compared to Merton's estimate of PD (i.e. $PD_{Merton} = N(-d_2)$), prediction power is far superior.
Moody’s starts by defining a “distance-to-default” $DD$:

$$DD = \ln \frac{A}{DP} + \left( \mu_A - \sigma_A^2/2 \right) \left( T - t \right) \frac{1}{\sigma_A \sqrt{T - t}}$$

which is just $d_2$ in the Merton model with $\bar{B}$ replaced with $DP$ and $r$ replaced with $\mu_A$. 

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$$DD = \ln \frac{A}{DP} + (\mu_A - \sigma_A^2/2)(T - t) \frac{\sigma_A \sqrt{T - t}}{\sigma_A \sqrt{T - t}}$$

which is just $d_2$ in the Merton model with $\bar{B}$ replaced with $DP$ and $r$ replaced with $\mu_A$.

- Default point $DP$ chosen as the short term debt plus half of long-term debt:

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- Then calibrate $DD$ to empirical distribution $\Psi$ of realized defaults from proprietary database:

$$PD_{Moody's} = \Psi(-DD)$$

- Compared to Merton’s estimate of PD (i.e. $PD_{Merton} = \mathcal{N}(-d_2)$), prediction power is far superior.
We assume

- Existence of a risk-neutral measure $\tilde{\mathbb{P}}$ with risk-free rate $r_f$.
- Default $\tau$ is adapted to the filtration $\mathcal{F}_t = \sigma(A_t, R_t)$.
- A recovery process $R_t$.
- Bond has maturity $T$.

It follows that

$$B_t(\omega) = \bar{B} e^{-r_f(T-t)} \tilde{\mathbb{P}}_t [\tau > T] + \tilde{\mathbb{E}}_t \left[ e^{-r_f(T-t)} R_T 1_{\{\tau = T\}} \right]$$

$$+ \tilde{\mathbb{E}}_t \left[ e^{-r_f(\tau-t)} R_\tau 1_{\{\tau < T\}} \right].$$

(3)
Complete Recovery Upon Default and No Default

If it happens that the entire arbitrage-free value of the bond is recovered upon default, i.e. under risk-free rate $r_f$ and risk-neutral measure $\tilde{P}$,

$$\tilde{E}_t [R_s] = B e^{-r_f(T-s)} \tag{4}$$

then the entire no-default value is retained:

$$B_t(\omega) = B e^{-r_f(T-t)} \tilde{P}_t [\tau > T] + \tilde{E}_t \left[ e^{-r_f(T-t)} B 1_{\{\tau = T\}} \right]$$

$$+ \int_t^T e^{-r_f(s-t)} B e^{-r_f(T-s)} \tilde{P}_t [\tau \in ds] = B e^{-r_f(T-t)} \tag{5}$$

Of course, if $\tilde{P}_t [\tau > T] = 1$, then $\tilde{P}_t [\tau \in ds] = 0$ for all $s \in [t, T]$ and so $B_t = B e^{-r_f(T-t)}$ once again.
Example-Basic Hazard Rate Model

As an initial example to motivate bond pricing formula, assume that both default time and recovery are independent of the Asset filtration, and default time is in fact memoryless:

- \( \tilde{P}_t[\tau > u] = e^{-\lambda(u-t)} \) for some \( \lambda > 0 \) and all \( u \geq t \geq 0 \).
- \( \tilde{E}_t[e^{-r_f s} R_s] = e^{-r_f t} R_t \).

It follows that

- \( B_t(\omega) = \bar{B} e^{-r_f (T-t)} e^{-\lambda(T-t)} + R_t(\omega) \cdot (1 - e^{-\lambda(T-t)}) \) and
- \( Y(t, T)(\omega) = -\frac{1}{T-t} \ln \left[ e^{-\lambda(T-t)} + \frac{R_t(\omega)}{\bar{B} e^{-r_f (T-t)}} (1 - e^{-\lambda(T-t)}) \right] \)
- \( \lim_{T \to t^+} Y(t, T)(\omega) = \lambda \left( 1 - \frac{R_t(\omega)}{\bar{B}} \right) \).

Note that the short term credit spread is positive a.e. as long as
\( \tilde{P}_t [R_t < \bar{B}] = 1 \).
Define the quantities:

- $A_t$, the asset value at time $t > 0$
- $R_t$, the recovery amount at time $t > 0$

We model recovery as a "shadow" asset, and thus follows the same dynamics as the asset.

The dynamics for the asset and recovery are, under a risk neutral measure $\tilde{P}$,

$$dA_t = rf_t dt + \sigma_A dW_A$$

$$dR_t = rf_t dt + \sigma_R dW_R.$$  \hspace{2cm} (6)

The risk drivers are correlated via

$$\tilde{E}[dW_A dW_R] = \rho A, R dt.$$ \hspace{2cm} (7)
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- $R_t$, the recovery amount at time $t > 0$. We model recovery as a "shadow" asset, and thus follows the same dynamics as the asset.

The dynamics for the asset and recovery are, under a risk neutral measure $\tilde{\mathbb{P}}$,

\[
\begin{align*}
\frac{dA_t}{A_t} &= r_f dt + \sigma_A dW_t^A \\
\frac{dR_t}{R_t} &= r_f dt + \sigma_R dW_t^R.
\end{align*}
\]  

(6)

The risk drivers are correlated via

\[\mathbb{E}[dW_t^A dW_t^R] = \rho_{A,R} dt.\]  

(7)
If we allow for a shadow recovery process as in equation (6), then we are left with an exchange option where the payoff at expiry is

\[ G(A_T, R_T) = R_T 1\{A_T < \bar{B}\} + \bar{B} 1\{A_T \geq \bar{B}\}. \]  (8)

Define

\[
\text{Default} = \{A_T \geq \bar{B}\}^c
\]

\[
\alpha := \delta + \frac{1}{2} \sigma_R^2 (1 - \rho_{A,R}^2) + \gamma \left( r_f - \frac{1}{2} \sigma_A^2 \right) + \frac{\gamma^2 \sigma_A^2}{2}
\]

\[
\gamma := \rho_{A,R} \frac{\sigma_R}{\sigma_A}
\]

\[
\delta := (r_f - \frac{1}{2} \sigma_R^2) - \gamma (r_f - \frac{1}{2} \sigma_A^2).
\]

Note that \(\rho_{A,R} = \gamma = 1\) implies that \(\delta = 0\) and \(\alpha = r_f\).
Consequently,

\[ B_t = e^{-r_f(T-t)}\hat{\mathbb{E}}_t \left[ R_T 1\{A_T < \bar{B}\} + \bar{B} 1\{A_T \geq \bar{B}\} \right] \]

\[ = \hat{\mathbb{E}}_t \left[ e^{-r_f(T-t)}R_T 1\{A_T < \bar{B}\} \right] + \bar{B}e^{-r_f(T-t)}\cdot \hat{\mathbb{P}}_t[A_T \geq \bar{B}] \]

\[ = R_t e^{(\alpha - r_f)(T-t)} \Phi \left( \frac{\ln \frac{\bar{B}}{A_t} - (r_f - \frac{1}{2}\sigma_A^2)(T-t) - \gamma\sigma_A^2(T-t)}{\sigma_A \sqrt{T-t}} \right) \]

\[ + \bar{B}e^{-r_f(T-t)} \cdot \left( 1 - \Phi \left( \frac{\ln \frac{\bar{B}}{A_t} - (r_f - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A \sqrt{T-t}} \right) \right) \]

Note that we retain the classical 1d Merton price if \( \rho_{A,R} = \gamma = 1 \).
Two Factor Merton Model - Default at Maturity

Recovery metrics:

\[
\mathbb{E}_t[\text{Loss} \mid \text{Default}] = \frac{\mathbb{E}_t[\bar{B} - R_T \mid A_T < \bar{B}]}{\bar{B}}
\]

\[
= 1 - \frac{R_t}{\bar{B}e^{-\alpha(T-t)}} \Phi \left( \frac{\ln \frac{\bar{B}}{A_t} - (r_f - \frac{1}{2}\sigma_A^2)(T-t) - \gamma \sigma_A^2(T-t)}{\sigma_A \sqrt{T-t}} \right)
\]

\[
\mathbb{P}_t[\text{Default}] = \Phi \left( \frac{\ln \frac{\bar{B}}{A_t} - (r_f - \frac{1}{2}\sigma_A^2)(T-t)}{\sigma_A \sqrt{T-t}} \right)
\]

\[
Y_{2D\ Merton}(t, T) = \frac{1}{T - t} \ln \left( \frac{1}{1 - \mathbb{P}_t[\text{Default}]\mathbb{E}_t[\text{Loss} \mid \text{Default}]} \right)
\]
Finally, it should be noted that since

\[ \lim_{\gamma \to 1} d_\gamma = d_1 \]

\[ \lim_{\rho \to 1, \gamma \to 1} \alpha = r_f \]  \hspace{1cm} (12)

\[ \lim_{\rho \to 1, \gamma \to 1, R_0 \to A_0} \tilde{\mathbb{P}}[A_t = R_t] = 1, \]

it follows that

\[ \lim_{\rho \to 1, \gamma \to 1, R_t \to A_t} Y_{2d \text{ Merton}}(t, T) = Y_{\text{Merton}}(t, T). \]  \hspace{1cm} (13)
Let $DP$ define our default point written into the bondholder covenant on the asset $A$, and define $\tau_{DP}$ as the first time $A$ reaches this default point.
Two Factor Black-Cox Model - Stopping Times

Let $DP$ define our default point written into the bondholder covenant on the asset $A$, and define $\tau_{DP}$ as the first time $A$ reaches this default point.

Furthermore, define $2dBC$ default time $\tau$ as

$$\tau = \begin{cases} 
\tau_{DP} & : \{\tau_{DP} < T\} \\
T & : \{\tau_{DP} \geq T\} \cap \{A_T < \bar{B}\} \\
\infty & : \text{otherwise.}
\end{cases}$$

Equivalently,

$$\text{Default} = \{\tau_{DP} > T, A_T \geq \bar{B}\}^c.$$  \hspace{1cm} (14)
Weak Covenant Model: $DP < \bar{B}$

**Theorem**

$$B_{t,T}^{2d \text{ Black Cox}} = \bar{B} e^{-r_f(T-t)} \cdot \tilde{P}_t[A_T \geq \bar{B}, \tau_{DP} > T]$$

$$+ \left[ R_t - \tilde{E}_t[e^{-r_f(T-t)} R_T 1\{A_T \geq \bar{B}, \tau_{DP} > T\}] \right]$$

$$:= \bar{B} e^{-r_f(T-t)} \left[ 1 - u(w, y, T-t) \right] + v(R_t, w, y, T-t)$$

$$y = \ln \frac{DP}{A_t} \leq 0$$

$$w = \ln \frac{\bar{B}}{A_t} \leq 0$$

$$\mu = r_f - \frac{1}{2} \sigma_A^2$$

$$\gamma = \rho_{A,R} \frac{\sigma_R}{\sigma_A}.$$
Weak Covenant Model: $DP < \bar{B}$

\[ u(w, y, T - t) = \mathcal{N}\left( \frac{w - \mu(T - t)}{\sqrt{\sigma^2_A(T - t)}} \right) + e^{\frac{2\mu y}{\sigma^2_A}} \mathcal{N}\left( \frac{-w - 2y - \mu(T - t)}{\sqrt{\sigma^2_A(T - t)}} \right) \]

\[ v(R_t, w, y, T - t) = R_t \mathcal{N}\left( \frac{w - (\mu + \gamma\sigma^2_A)(T - t)}{\sqrt{\sigma^2_A(T - t)}} \right) + R_t e^{2\gamma y + \frac{2\mu y}{\sigma^2_A}} \mathcal{N}\left( \frac{-w - 2y - (\mu + \gamma\sigma^2_A)(T - t)}{\sqrt{\sigma^2_A(T - t)}} \right) \]

(16)
We can also complete this calculation using the fact that $e^{-rf_t R_t}$ is a local martingale, and so for the bounded stopping time $\tau := \min \{ \tau_{DP}, T \}$,

$$e^{-rf_t R_t} = \tilde{E}_t[e^{-rf_T R_T}] = \tilde{E}_t[e^{-rf_T R_T 1_{\{\tau \leq T\}}} + \tilde{E}_t[e^{-rf_T R_T 1_{\{\tau > T\}}}]$$

It follows that

$$B_t = R_t + \tilde{B}e^{-rf(T-t)} \cdot \tilde{P}_t[A_T \geq \tilde{B}, \tau_{DP} > T]$$

$$- \frac{R_t}{A_t^\gamma} e^{-(rf-\delta-\frac{1}{2} \sigma_R^2 (1-\rho_{A,R}^2)) (T-t)} \cdot \int_{\tau_{DP}}^\infty a^\gamma \tilde{P}_t[A_T \in da, \tau_{DP} > T]$$
We calculate the remaining integral using the identities from the Double Lookbacks paper. We begin with a standard Brownian motion $W$ on a probability space and define

\[ X_t = \mu t + \sigma W_t \]
\[ \tau_a = \min \{ t : X_t = a \} \]
\[ \underline{X}_t = \min_{0 \leq s \leq t} X_s \]

\[ g(x, y, t, \mu) := \frac{1}{\sigma \sqrt{t}} \phi\left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) \left( 1 - \exp\left( \frac{4y^2 - 4xy \sigma^2}{2\sigma^2 t} \right) \right) \] 

(19)
The authors of the Double Lookback paper prove that

\[
\tilde{P} [X_t \in dX, X_t \geq y] = g(-x, -y, t, -\mu) dx
\]

\[
= \frac{1}{\sigma \sqrt{t}} \phi\left( -\frac{x + \mu t}{\sigma \sqrt{t}} \right) \left( 1 - \exp \left( \frac{4y^2 - 4xy}{2\sigma^2 t} \right) \right)
\]

\[
= \frac{1}{\sigma \sqrt{t}} \phi\left( -\frac{(x - \mu t)}{\sigma \sqrt{t}} \right) \left( 1 - \exp \left( - \frac{4y(x - y)}{2\sigma^2 t} \right) \right)
\]

(20)
By setting

\[ \mu := r_f - \frac{1}{2}\sigma_A^2 \]  \hspace{1cm} (21)

\[ \sigma := \sigma_A \]

it follows that

\[ X_t = \ln \frac{A_t}{A_0} \]

\[ X_t = \min_{0 \leq s \leq t} \ln \frac{A_s}{A_0} \]  \hspace{1cm} (22)

and so for \( y = \ln \frac{DP}{A_0} \),

\[ A_t = A_0 e^{X_t} \]

\[ \{ \tau_{DP} > T \} = \left\{ X_T > \ln \left( \frac{DP}{A_0} \right) \right\} = \left\{ \tau_y^X > T \right\}. \]  \hspace{1cm} (23)
Using this notation, we combine the above to retain,

\[
\int_{DP}^\infty a^\gamma \cdot \tilde{P}_t \left[ A_T \in da, \tau_{DP} > T \right] = \tilde{E}_t \left[ A_T^\gamma 1\{\tau_{DP} > T\} \right] \\
= \tilde{E}_0 \left[ e^{\gamma \left( \ln(A_t) + X_{T-t} \right)} 1\{\tau^X_y > T-t\} \right] \\
= \int_y^\infty e^{\gamma (x + \ln A_t)} \tilde{P}_0 \left[ X_{T-t} \in dx, \tau^X_y > T-t \right] \\
= (A_t)^\gamma \int_y^\infty \frac{1}{\sqrt{2\pi \sigma^2_A (T-t)}} e^{\left( -\frac{(x-\mu(T-t))^2 - 2\sigma^2_A \gamma (T-t)}{2\sigma^2_A (T-t)} \right)} dx \\
- (A_t)^\gamma \int_y^\infty \frac{1}{\sqrt{2\pi \sigma^2_A (T-t)}} e^{\left( -\frac{(x-\mu(T-t))^2 + 4y(x-y) - 2\sigma^2_A \gamma (T-t)}{2\sigma^2_A (T-t)} \right)} dx.
\]

(24)
Consistency: Reduction to 1-D Black-Cox Model

As a result of the closed form above for $B_{t,T}$, the consistency of the model as $T \to t$ follows quickly:

**Lemma**

$$\lim_{T \to t^+} B_{t,T}^{2d \text{ Black Cox}} = \bar{B}.$$ 

**Proof.**

Without loss of generality, we assume $t = 0$. As $y, w < 0$,

$$\lim_{T \to 0^+} \left[ N \left( -\frac{w - \mu T}{\sqrt{\sigma^2_A T}} \right) - e^{\frac{2\mu y}{\sigma^2_A}} N \left( -\frac{w - 2y - \mu T}{\sqrt{\sigma^2_A T}} \right) \right] = 1$$

$$\lim_{T \to 0^+} \left[ N \left( \frac{w - (\mu + \gamma \sigma^2_A) T}{\sqrt{\sigma^2_A T}} \right) + e^{2\gamma y + \frac{2\mu y}{\sigma^2_A}} N \left( -\frac{w - 2y - (\mu + \gamma \sigma^2_A) T}{\sqrt{\sigma^2_A T}} \right) \right] = 0.$$
Consistency: Reduction to 1-D Black-Cox Model

\[ B^{1d\ \text{Black-Cox}}_t, T = \lim_{(R_t, \rho_{A,R}, \sigma_R) \to (A_t, 1, \sigma_A)} B^{2d\ \text{Black-Cox}}_t, T \]

\[ = \bar{B} e^{-rf(T-t)} N \left( -\ln \frac{\bar{B}}{A_t} - \mu^{-}(T-t) \right) \sqrt{\sigma^2_A(T-t)} \sqrt{\sigma^2_A(T-t)} \]

\[ = \bar{B} e^{-rf(T-t)} \left( \frac{DP}{A_t} \right)^{\frac{2\mu^{-}}{\sigma^2_A}} N \left( -\ln \frac{\bar{B}A_t}{DP^2} - \mu^{-}(T-t) \right) \sqrt{\sigma^2_A(T-t)} \sqrt{\sigma^2_A(T-t)} \]

\[ + A_t \left[ N \left( -\ln \frac{\bar{B}}{A_t} - \mu^{-}(T-t) \right) \sqrt{\sigma^2_A(T-t)} \right] + \left( \frac{DP}{A_t} \right)^{\frac{2\mu^{+}}{\sigma^2_A}} N \left( -\ln \frac{\bar{B}A_t}{DP^2} - \mu^{-}(T-t) \right) \sqrt{\sigma^2_A(T-t)} \sqrt{\sigma^2_A(T-t)} \right]. \]

\[ \mu^{+} = rf + \frac{1}{2} \sigma^2_A \]

\[ \mu^{-} = rf - \frac{1}{2} \sigma^2_A. \]
Credit Spread: 2-D Black-Cox Model

By appealing to the martingale nature of recovery, and Bayes’ Theorem,

\[
Y_{2dBC}(t, T) = \frac{1}{T - t} \ln \left( \frac{1}{1 - \mathbb{E}_t[\text{Loss} \mid \text{Default}] \cdot \tilde{\mathbb{P}}_t[\{\text{Default}\}]} \right)
\]

\[
\mathbb{E}_t[\text{Loss} \mid \text{Default}] = \mathbb{E}_t\left[ \frac{\bar{B} - R_T}{\bar{B}} \mid \text{Default} \right]
\]

\[
= 1 - \frac{\mathbb{E}_t[e^{-r_f(T-t)}R_T \mid \text{Default}]}{\bar{B}e^{-r_f(T-t)}}
\]

\[
= 1 - \frac{R_t - \mathbb{E}_t[e^{-r_f(T-t)}R_T 1\{\text{No Default}\}]}{\bar{B}e^{-r_f(T-t)}\tilde{\mathbb{P}}_t[\text{Default}]} \tilde{\mathbb{P}}_t[\text{Default}]
\]

\[
= 1 - \frac{v(R_t, w, y, T - t)}{\bar{B}e^{-r_f(T-t)}[1 - u(w, y, T - t)]}
\]

\[
\tilde{\mathbb{P}}_t[\text{Default}] = 1 - u(w, y, T - t) = PD_{t,T}^{1dBC}. \tag{27}
\]
Without loss of generality, we can define the CDS Premium at time $0$. Under the usual (EPP) framework for continuous payment, premium rate $P_{0,T}$

$$
\tilde{E}_0[e^{-rf \tau_{DP}} (\bar{B} - R_{\tau_{DP}}) 1_{\{\tau_{DP} \leq T\}}] + \tilde{E}_0[e^{-rf T} (\bar{B} - R_T) 1_{\{A_T < \bar{B}, \tau_{DP} > T\}}] \\
= \int_0^T e^{-rf s} \tilde{P}_0[\tau_{DP} > s] ds
$$

$$
\tilde{E}_0[e^{-rf \tau_{DP}} \bar{B} 1_{\{\tau_{DP} \leq T\}}] + \tilde{E}_0[e^{-rf T} \bar{B} 1_{\{A_T < \bar{B}, \tau_{DP} > T\}}] - v(R_0, w, y, T) \\
= \int_0^T e^{-rf s} \tilde{P}_0[\tau_{DP} > s] ds
$$

$$
\bar{B} \left( \tilde{E}_0[e^{-rf \tau_{DP}} 1_{\{\tau_{DP} \leq T\}}] + e^{-rf T} \tilde{P}_0[A_T < \bar{B}, \tau_{DP} > T] \right) - v(R_0, w, y, T) \\
= \int_0^T e^{-rf s} \tilde{P}_0[\tau_{DP} > s] ds
$$

(28)
It follows that

\[ P_{0,T} = \frac{\bar{B} \left( h(A_0, T) + e^{-rf T} [F(T) - u(w, y, T)] \right) - v(R_0, w, y, T)}{\int_0^T e^{-rf s} F(s) ds} \]

\[ = r_f \frac{\bar{B} \left( h(A_0, T) + e^{-rf T} [F(T) - u(w, y, T)] \right) - v(R_0, w, y, T)}{1 - e^{-rf T} F(T) - h(A_0, T)} \]
CDS: 2-D Black-Cox Model

It follows that

\[ P_{0, T} = e^{-r_f T} \left[ B \left( h(A_0, T) + e^{-r_f T} [F(T) - u(w, y, T)] \right) - \nu(R_0, w, y, T) \right] - \nu(R_0, w, y, T) \int_0^T e^{-r_f s} F(s) ds \]

\[ = r_f \left( \frac{B \left( h(A_0, T) + e^{-r_f T} [F(T) - u(w, y, T)] \right) - \nu(R_0, w, y, T)}{1 - e^{-r_f T} F(T) - h(A_0, T)} \right) \]

\[ F(s) = N \left( \frac{-w + \mu_s}{\sqrt{\sigma_A^2 s}} \right) - e^{\frac{2\mu_y}{\sigma_A^2}} N \left( \frac{-w + 2y + \mu_s}{\sqrt{\sigma_A^2 s}} \right) \]

\[ h(A_0, T) = \tilde{E}_0 \left[ e^{-r_f \tau_{DP}} 1 \{ \tau_{DP} \leq T \} \right] \]

\[ = A_0 \left[ N \left( \frac{\ln \frac{\tilde{B}}{A_0} - \left( r_f + \frac{1}{2} \sigma_A^2 \right) T}{\sqrt{\sigma_A^2 T}} \right) - \left( \frac{DP}{A_0} \right) \frac{2(r_f + \frac{1}{2} \sigma_A^2)}{\sigma_A^2} N \left( \frac{2 \ln \frac{DP}{A_0} - \ln \frac{\tilde{B}}{A_0} + \left( r_f + \frac{1}{2} \sigma_A^2 \right) T}{\sqrt{\sigma_A^2 T}} \right) \right] \]

(29)
Conclusions and Next Papers?

- Have extended 2D models to include closed form solutions (both Merton and Black-Cox).
- Recovery is unbounded. Historically possible (foreclosure and bank fees.)
- Can include bounded recovery in this framework, work underway.
- Can extend to other products, such as Credit Linked Notes.
- Calibration to market data for both CDS and Bond prices (any takers?)
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Where it all started!! Albert and Nick (circa 1999)