Dolgachev surface

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4-manifolds are spaces that locally look like $\mathbb{R}^4$
4-Manifolds, as you see it

he lives in 4-manifold!
k-handle $B^4 = B^k \times B^{4-k}$ attached along $S^{k-1} \times B^{4-k}$

You only need to understand this much .. but

These 3-handles at top makes our understanding harder!

attaching is the same as drilling!
Dolgachev surface $E(1)_{2,3}$ is an elliptic complex surface, which is obtained from the standard elliptic surface $E(1) = \mathbb{C}P^2 \# 9 \overline{\mathbb{C}P}^2$, by the log transform operation of orders 2 and 3 on two disjoint fibers.

$E(1)$: Let $P_0(z)$ and $P_1(z)$ be a generic pair of homogeneous cubic polynomials in $\mathbb{C}^3$. For each $t = [t_0, t_1] \in \mathbb{C}P^1$ the following sets $Z_t = \{z \in \mathbb{C}P^2 \mid t_0 P_0(z) + t_1 P_1(z) = 0\}$ fill $\mathbb{C}P^2$ (generically each $Z_t$ is a torus).
$E(1)_{2,3}$

- $E(1)_{2,3}$ is obtained from the standard elliptic surface $E(1)$, by the log transform operation of orders $p = 2$ and $p = 3$ on two disjoint fibers. Remove two disjoint fibers $T^2 \times D^2$ and glue back by the diffeomorphism of $T^3$ given by

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & p
\end{pmatrix}$$

- Equivalently $E(1)_{2,3}$ is obtained by replacing one $T^2 \times D^2$ by $(S^3 - N(K)) \times S^1$, where $N(K)$ is the tublar neighborhood of the trefoil knot in $S^3$ (so called "Knot Surgery" operation of Fint-Stern)
Is there an exotic copy of $S^4$ or $\mathbb{CP}^2$?

- If an exotic copy of $S^4$ or $\mathbb{CP}^2$ exits it must have 1- or 3- handles!
- 24 years ago Donaldson gave the first example of an oriented exotic smooth 4-manifold. He proved that Dolgachev surface $E(1)_{2,3}$ is an exotic copy of $E(1)$. About the same time, Harer, Kas and Kirby wrote a book about $E(1)_{2,3}$ where they conjectured that it must contain 1- or 3- handles.
- (A) In 2008 Kirby-Kas-Harer conjecture was disproved: $E(1)_{2,3}$ admits an handlebody without 1- and 3-handles. To prove this we start with an handle picture of $E(1)_{2,3}$, and turn this handlebody upside down twice while canceling its handles! What does this say about the exotic smooth structures on 4 manifolds?
First cancel 1-handles, in order to cancel 3-handles turn it upside down and cancel 1-handles. Finally to make it look pretty turn it upside down again. Keep turning it upside down as you cancel its handles.
Here is the picture of $E(1)_{2,3}$ without 1- or 3-handles

How does this exotic copy of $E(1)$ compare with its original?
How does an exotic copy of any smooth $M$ look like?

Let $M$ be a smooth closed simply connected 4-manifold, and $M'$ be an exotic copy of $M$. Then we can find a compact contractible codim zero submanifold $W \subset M$ with complement $N$, and an involution $f : \partial W \to \partial W$ giving decompositions: $M = N \cup_{id} W$, $M' = N \cup_{f} W$

Furthermore, we can make the each piece $W$ and $N$ Stein manifolds!

(This was first observed on an example by A, then the general result was proven by Matveyev and independently by Curtis-Freedman-Hsiang-Stong. The Stein part is due to A and Matveyev.)
Corks

- **Cork** is a pair $(\mathcal{W}, f)$, where $\mathcal{W}$ is a compact contractible Stein manifold, and $f : \partial \mathcal{W} \to \partial \mathcal{W}$ is an involution, which extends to a self-homeomorphism of $\mathcal{W}$, but it does not extend to a self-diffeomorphism of $\mathcal{W}$. We say $(\mathcal{W}, f)$ is a cork of $M$, if we have the above decomposition for some exotic copy $M'$ of $M$.

![Diagram of Cork with involution](image)
Only when I am standing on it, this space looks exotic!

flipping the boundary

exotic copy of W rel boundary
Wallpaper depiction of a manifold and its exotic copy

standard manifold $M$

absolute exotic copy of $M$
Various corks

\[ W_1 \]

\[ W_n \]

\[ \bar{W}_n \]
(A) 1991 (where the first “cork” was introduced)
Locating corks in the small exotic 4-manifold pairs

\[ W^{-1} = \text{id} \]

\[ W = \text{single 2-handle} \]

\[ \text{id} = \begin{cases} H & \text{for } W \end{cases} \]

\[ f(\alpha) = \begin{cases} H & \text{for } W \end{cases} \]
Amusing diffeomorphic 4-manifold pairs
(not everything which looks exotic is exotic!)

(A) 1977 (where “circle with dot ” notation for 1-handle introduced)
A Plug is a pair \((W, f)\), where \(W\) is a compact Stein manifold, and \(f : \partial W \to \partial W\) is an involution, which does not extend to a self homeomorphism of \(W\), and there is the above decomposition for some exotic copy \(M'\) of \(M\) (*Plugs were discovered by A and K. Yasui*).

\[
\begin{align*}
\text{the nontrivial} & \quad \text{diffeomorphism of} \\
S^1 \times S^2 & \\
\end{align*}
\]

\[
\begin{align*}
f & \quad = \\
0 & \\
\end{align*}
\]

\[
\begin{align*}
\text{degenerate plug} & \\
B^4 & \\
\end{align*}
\]

\[
\begin{align*}
4\text{-ball with 2-handle} & \quad \text{attached to the unknot} \quad \text{with framing 0} \\
& \\
\end{align*}
\]

\[
\begin{align*}
W_{m,n} & \quad f \quad (\alpha) \\
0 & \\
\end{align*}
\]

\[
\begin{align*}
\text{plug} & \\
B^4 & \\
\end{align*}
\]

\[
\begin{align*}
4\text{-ball with 2-handle} & \quad \text{attached to the torus knot} \quad (n, nm+1) \quad \text{with framing} \quad -m-2n \\
& \\
\end{align*}
\]
Corks and plugs should be considered as freely moving basic particles in a 4-manifold $M$ relating it to its exotic copies. Corks and Plugs can knot in $M$ infinitely many different ways (A-Yasui). Well then, where are the corks/plugs of $E(1)_{2,3}$ relating it to $E(1)$?
Cork and Plug decompositions of $E(1)_{2,3}$

(1) $E(1)_{2,3}$ is obtained by cork twisting of $E(1)$ along the cork $ar{W}_1$, i.e. we can decompose $E(1)_{2,3} = N \cup_{id} \bar{W}_1$, so that $E(1) = N \cup_f \bar{W}_1$.

(2) $E(1)_{2,3}$ is obtained by plug twisting of $E(1)$ along the plug $W_{1,2}$, we can decompose $E(1)_{2,3} = N \cup_{id} W_{1,2}$, with $E(1) = N \cup_f W_{1,2}$.

(3) $E(1)_{2,3}$ is obtained from $E(1)$ by twisting along an $RP^2$.

Remarks: (1) is proven by inspection, (2) uses upside down turning technique, plus the result "Scharlemann manifold is standard". (3) follows from (2) since $W_{1,2}$ contains an $RP^2$. 

\[\begin{align*}
\text{Remarks: } (1) & \text{ is proven by inspection, (2) uses upside down turning technique, plus the result } \text{"Scharlemann manifold is standard". (3) follows from (2) since } W_{1,2} \text{ contains an } RP^2.
\end{align*}\]
From inside 4-manifold watching a cork/plug in the sky

Plug = $B^4 + 2$-handle

from inside the plug handles $a$ and $f(a)$ will look far from each other
Locating the $W_{1,2}$ plug in $E(1)_{2,3}$ (a road map)

To prove that this part remains $S^2 \times S^2$ after knot surgery, we use the Theorem: "Scharlemann’s manifold is standard"