Casson's Invariant for Oriented Homology 3-Spheres

An Exposition

by

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Preface

In the spring of 1985, A. Casson announced an interesting invariant of homology 3-spheres via beautiful constructions on representation spaces. This invariant generalizes the Rohlin invariant and gives surprising corollaries in low dimensional topology. Armed with the Casson lecture notes and many informal discussions with him, in the fall of 1985 we held a seminar on this invariant. These notes grew out of this seminar. We tried to remain close to Casson's original outline and proceeded by giving needed details including an exposition of Newstead's results. We have often chosen classical concrete approaches over general methods; this perhaps made this exposition at places longer and more elementary than necessary. For example, we did not attempt to give gauge theory explanations for the results of Newstead. Instead, we followed his original techniques. We would like to thank the seminar participants at M.S.U. and Tammy Halffield and Cathy Friese for a speedy typing of these notes.
Introduction

Let $M^3$ be an oriented homology 3-sphere and $K$ be a knot in $M^3$. Let $N(K)$ be a closed regular neighborhood of $K$, so that $N(K) \cong S^1 \times D^2$. Let $M^3(K)$ be the complement of the interior of $N(K)$ in $M^3$. $M^3(K)$ is a 3-manifold with boundary. The boundary is a torus $T^2$. Since $M^3$ is a homology 3-sphere, we can choose the identification of $N(K)$ with $S^1 \times D^2$ so that $S^1 \times \{1\}$ is homologically trivial in $M^3(K)$. Under this assumption, let $\mu$ be the meridian $\{1\} \times 2D^2$ and $\lambda$ be the preferred longitude $S^1 \times \{1\}$. Since $M^3$ is oriented, $T^2$ inherits a natural orientation from the induced orientation on $M^3(K)$. We may orient the curves $\mu$ and $\lambda$ so that $(\mu, \lambda)$ is a positively oriented basis for the first homology of $T^2 \cong H_1(T^2)$.

Now for each integer $n$, the homology class $\mu + n\lambda$ can be represented by a simple closed curve $\gamma$ in $T^2$. We may obtain a new oriented homology 3-sphere by attaching $S^1 \times D^2$ to $M^3(K)$ along their respective boundaries in such a manner that $\{1\} \times 2D^2$ is attached to $\gamma$. The resulting oriented homology 3-sphere $K_n$ is well defined up to orientation preserving homeomorphism. We say that $K_n$ is the manifold obtained from $M$ by $1/n$-surgery on $K$. In particular, $K_0$ is $M$.

As for knots in the 3-sphere $S^3$, one can associate an Alexander-polynomial to $K$. This is a polynomial in the variables $t$ and $t^{-1}$ with integer coefficients and is well-defined up to multiplication by $t^n$ for any integer $n$. Since this polynomial satisfies the standard identities for Alexander polynomials of knots in $S^3$, one obtains a well defined polynomial, known as the symmetrized Alexander polynomial $\Delta_G$ by imposing the conditions:
\[ \Delta_R(t) = \frac{\Delta_R(t^{-1})}{\Delta_R(1)} \quad \Delta_R(1) = 1. \]

It is easy to see that these conditions imply that the second derivative of the Alexander polynomial evaluated at 1 \( \Delta_R'(1) \) is an even integer.

With the above notation, we may state Casson's main theorem concerning his new invariant for oriented homology spheres.

**Theorem (Casson):** There is a unique invariant, \( \lambda(M) \), of oriented homology 3-spheres satisfying the following properties:

1. \( \lambda(K_{R+1}) - \lambda(K_R) = \frac{1}{2} \Delta_R'(1) \) for any knot \( K \) in any oriented homology 2-sphere \( M^3 \).
2. \( \lambda(S^3) = 0 \).

The idea of Casson's invariant comes from the observation that if \( M^3 \) is a 3-manifold with a Heegaard splitting \( M^3 = W_1 \cup W_2 \), (where \( W_1 \) and \( W_2 \) are solid handlebodies of genus \( g \) meeting along their common boundary \( F \)), then the commutative diagram of inclusions:

\[
\begin{array}{ccc}
\pi_1(F) & \longrightarrow & \pi_1(W_1) \\
\downarrow & & \downarrow \\
\pi_1(W_2) & \longrightarrow & \pi_1(M^3)
\end{array}
\]

gives rise to imbeddings of the space of conjugacy classes of representations of these groups into \( SU(2) \):

\[
\begin{array}{ccc}
R(F) & \longrightarrow & R(W_1) \\
\downarrow & & \downarrow \\
R(W_2) & \longrightarrow & R(M)
\end{array}
\]

It turns out that all these spaces are real algebraic sets and \( R(W_i) \) and \( R(W_j) \) are complementary dimensional subspaces of \( R(F) \). The singular set \( \Sigma_i \) of \( R(W_i) \) is included in the singular set \( \Sigma \) of \( R(F) \). Casson's invariant is roughly the "algebraic intersection number" of the smooth manifolds \( R(W_i) \) in \( R(F) \). The technical difficulties are to make sense out of the algebraic intersection number of these proper open submanifolds and to show that it is independent of the Heegaard splitting of \( M \).

The properties (1) and (2) actually allow us to compute \( \lambda(M) \) for any homology sphere. This may be done as follows. Using a well known result ([L]), we may represent \( M \) by surgery on a framed link in \( S^3 \) (with integer framings). Moreover, we can assume that the linking matrix is unimodular, odd and indefinite. With this assumption, it is possible, by handle sliding, to diagonalize the matrix. Hence, we find that \( M \) is given by
surgery on a framed link in $S^3$ with a diagonal linking matrix whose diagonal entries are all $\pm 1$.

This implies that we can obtain $M^3$ by a sequence of $\pm 1$ Dehn surgeries on knots in homology spheres beginning with a knot in $S^3$. Hence, we can calculate $\lambda(M)$ inductively by applying property (1) beginning with property (2). In particular, $\lambda(M)$ is an integer.

This observation establishes the uniqueness of $\lambda$. Hence, it remains to establish existence. This task will occupy our attention throughout this exposition. In the course of proving existence, it is found that Casson's invariant satisfies the following additional properties.

**Theorem (Casson):** The invariant $\lambda$ satisfies the following properties:

1. $\lambda(-M) = -\lambda(M)$.
2. $\lambda(M) = \mu(M)$ (mod 2), where $\mu(M)$ is the Rohlin invariant.
3. $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.

From our description of calculating $\lambda$, it is easy to see that properties (1) and (2) imply properties (3) and (5). Property (5) also follows from properties (1) and (2) and known relations between the Rohlin invariant of 3-manifolds obtained by surgery to a knot $K$, the Arf invariant of $K$ and the Alexander polynomial of $K$ (see section V.8(c)).

Property (4), as we shall see, follows immediately from the definition of $\lambda$. Representations of $\Pi_1(M)$ in $SU(2,\mathbb{C})$ form the content for $\lambda$ in Casson's formulation of this theory. As Casson suggested in his lectures on the summer of 1985, and Taubes latter verified, there is an alternative approach to this invariant via the gauge theory of flat connections on principal $SU(2,\mathbb{C})$-bundles. This links Casson's work with the developments of Donaldson in 4-manifolds in a way that is yet to be fully understood.

We now list some immediate corollaries of the properties of $\lambda$, ($\lambda'(K)$ will denote $\lambda(K_{n+1}) - \lambda(K_n)$).

**Corollary (Casson):** Any homotopy 3-sphere has zero Rohlin invariant.

**Corollary (Casson):** Any amphicheiral homology 3-sphere has zero Rohlin invariant.

**Corollary (Casson):** If $\lambda'(K)$ is nonzero and $K$ is in $S^3$, then $K$ has property P.

**Proof:** From property (1), we deduce that:

$$\lambda(K_{n+1}) - \lambda(K_n) = n \cdot \lambda'(K).$$

Since $K_0 = S^3$, by property (2), we have:

$$\lambda(K_0) = n \cdot \lambda'(K).$$

Hence, by property (2) and our assumption on $\lambda'(K)$, we conclude that

$$K_n = S^3$$

if and only if $n$ is zero. \(\Box\)

Casson observed in his lectures that $\lambda$ is not an invariant of homology cobordism. There exist homology 3-spheres which bound smooth contractible 4-manifolds but have nonzero $\lambda$. 
Corollary (Casson): There exist non-triangulable 4-manifolds. For example, Freedman's $E_8$ manifold cannot be triangulated as a simplicial complex.

This is because, if $M^4$ were triangulated it would be P.L. in the complement of the open star of a vertex $x_0 \in M^4$. This would give the $\text{Link}(x_0)$ a P.L. structure $\mathbb{R}^3$. Since $M^4$ is a topological manifold, $\mathbb{R}^3$ has to be simply connected. So it must have zero Casson invariant, hence, zero Rohlin invariant $\mu(\Sigma)$. On the other hand, $\mathbb{R}^3$ is the boundary of the Spin 4-manifold $W^4 = M^4 \text{-open star}(x_0)$ and, therefore, $\mu(\Sigma) = 1/8 \text{signature}(W) \mod 2 = 1$, giving us a contradiction. We must add that in dimension 3 and 4 P.L. manifolds have smooth structures.

We close this introduction by giving an outline of the subsequent exposition.

Chapters are labeled by Roman numerals. Sections of a chapter are labeled by standard numerals. Subsections of a section are labeled by lowercase letters. Significant statements are labeled consecutively within each section. Hence, section V.2(c) refers to the third subsection of the second section of the fifth chapter. Likewise, Proposition IV.2.1 refers to the first proposition of the second section of the fourth chapter. Within each chapter, the reference to the chapter is omitted. Hence, a reference to Proposition 5.1 in the text of chapter IV refers to the first proposition in the fifth section of that chapter.

Chapters I and II are background material, concerning, respectively, the theory of representation spaces and the theory of Heegaard decompositions, two essential tools to Casson's approach. The results here, especially those in chapter II are well known propositions which the reader may wish to consult at various points in the subsequent chapters.

Chapter III is also background information concerning the particular representation spaces associated to an Heegaard decomposition of an homology sphere.

Chapter IV concerns the definition of Casson's invariant and the proof of its invariance (independence of Heegaard decomposition). Properties (2) and (4) are obvious from this definition.

Chapter V is a study of the associated invariant for knots in homology spheres $\lambda'(K)$. In this chapter, we establish the crucial axiomatic property (1). This involves a careful study of a "characteristic" cycle associated to the difference $\lambda(K_{n+1}) - \lambda(K_n)$, the difference cycle $\delta$. This cycle plays a central role in the exposition. (It may be of some interest to determine whether there are other "characteristic" cycles present in these representation spaces.)

In chapter VI, we study the topology of an associated representation space which arises in studying $\delta$. The main purpose of this section is to prove that $\delta$ is invariant under the Torelli group, a technical fact needed in chapter V. The main content of this section is extracted from the articles of Newstead.

One final disclaimer. The purpose of this text is purely expository. We have not made a systematic attempt to cross-reference the various sources consulted. The major portion of the exposition is as elementary as possible. We hope that the reader will find the exposition helpful and that the brief list
CHAPTER I: REPRESENTATION SPACES

1. The special unitary group - SU(2, $\mathbb{C}$).
   (a) Identification with $S^3$

   Let
   \[ M(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\} \]
   \[ SU(2, \mathbb{C}) = \{ A \in M(2, \mathbb{C}) \mid A^* A = I, \det(A) = 1 \} \]
   \[ S^3 = \{ (a, b) \in \mathbb{C}^3 \mid a \bar{a} + b \bar{b} = 1 \} \]

   There is a natural identification of $S^3$ with $SU(2, \mathbb{C})$:

   \[ S^3 \xrightarrow{\sim} SU(2, \mathbb{C}) \]
   \[ (a, b) \mapsto \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \]

   Conventions:

   (1) We shall identify $S^3$ and $SU(2, \mathbb{C})$ by this fixed diffeomorphism.

   (2) We shall consider $S^3$ to have a fixed orientation throughout our discussion.

   (b) The tangent bundle of $S^3$ $TS^3$

   By differentiating the defining equations for $S^3$, we obtain the tangent bundle of $S^3$: 
\[ TS^3 = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \begin{bmatrix} u & v \\ -\overline{v} & \overline{u} \end{bmatrix} \mid au + \overline{a}u + bv + \overline{b}v = 0, \overline{a}a + b\overline{b} = 1. \]

In particular, the Lie algebra of \( S^3 \) is given as:

\[ g = SU(2,\mathbb{C}) = SU(2,\mathbb{C}). \]

\[ G = \begin{bmatrix} is & v \\ -\overline{v} & 1 & \end{bmatrix} \mid s \in \mathbb{R}, v \in \mathbb{C}. \]

The action of \( S^3 \) on itself by left translations provides a trivialization of \( TS^3 \):

\[ S^3 \times S^3 \to TS^3 \]

\[ (A,X) \to (A,A\cdot X). \]

Convention: We identify \( TS^3 \) with \( S^3 \times S^3 \) by the natural trivialization provided above.

A natural Riemannian metric arises via translation from the inner product on \( S^3 \):

\[ \langle \cdot, \cdot \rangle : S^3 \times S^3 \to \mathbb{R} \]

\[ (X,Y) \to \frac{1}{2} \text{tr}(XY^t). \]

Associated to this metric, we have a natural exponential map:

\[ \exp : S^3 \to S^3. \]

The natural identification of \( S^3 \) with \( SU(2,\mathbb{C}) \) is an isometry of the standard metric on \( S^3 \) and the above metric on \( SU(2,\mathbb{C}). \)

(c) The conjugation actions on \( S^3 \) and \( TS^3 \)

\( S^3 \) acts on \( S^3 \) by conjugation:

\[ : S^3 \times S^3 \to S^3 \]

\[ (A,B) \to ABA^{-1}. \]

The derivative of this action is simply:

\[ : S^3 \times TS^3 \to TS^3 \]

\[ (A,(B,X)) \to (ABA^{-1},AXA^{-1}). \]

From the definition of the inner product on \( S^3 \):

\[ \langle AXA^{-1},AYA^{-1} \rangle = \langle X,Y \rangle. \]

Hence, these actions are actions by isometries and:

\[ \exp(AXA^{-1}) = A \exp(X)A^{-1}. \]

Let:

\[ S^0 = \{A\} \]

\[ S^1 = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \mid \theta \in \mathbb{R}. \]

Every element of \( S^3 \) is conjugate to a diagonal element:

**Proposition 1.1:** (1) \( S^0 \subset S^1 \subset S^3 \)

(2) \( S^3 = v \{CS^1C^{-1} \mid C \in S^3\} \)

**Proof:** (1) This is, of course, obvious. It is stated in anticipation of an analogous statement.
(2) Suppose $A \in S^3$. Since $\mathbb{C}$ is algebraically closed, $A$ has an eigenspace of eigenvalue $\lambda$. Choose a unit eigenvector in this eigenspace:

$$(x_0, y_0) \in S^2 \subset \mathbb{C}^2$$

$$V = \{ (x, y) \in \mathbb{C}^2 \mid (x, y) = \lambda (x_0, y_0) \} \subset \mathbb{C}$$

$$AV = \lambda V.$$ 

Let $C = \begin{bmatrix} x_0 & -y_0 \\ y_0 & x_0 \end{bmatrix}$, so that:

$$C^{-1} AC \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}, \text{ or } C^{-1} AC = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}.$$ 

Since $C \in SU(2, \mathbb{C})$,

$$C^{-1} AC = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \quad \lambda \bar{\lambda} = 1.$$ 

That is to say, $A \in \mathbb{C} S^1 C^{-1}$. \hfill \Box 

From the previous facts and the fact that:

$$\exp \begin{bmatrix} i\theta & 0 \\ 0 & -i\theta \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

we deduce that the exponential map is a diffeomorphism on the ball of radius $\pi$ about $0$:

$$\exp : B_\pi(0) \rightarrow B_\pi(0).$$

(Clearly, the injectivity radius of $S^3$ is $\pi$.)

Notation: We shall use the following abuses of notation:

$$i\theta = \begin{bmatrix} i\theta & 0 \\ 0 & -i\theta \end{bmatrix}, \quad e^{i\theta} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$ 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{}
\end{figure}

In particular, the natural contraction of $B_\pi(0)$:

$$G : B_\pi(0) \times [0,1] \rightarrow B_\pi(0)$$

$$(X, t) \mapsto tX.$$ 

exponentiates to a natural contraction of $S^3 \setminus \{-I\}$:

$$\sigma : (S^3 \setminus \{-I\}) \times [0,1] \rightarrow S^3 \setminus \{-I\}$$

$$(A, t) \mapsto A^t.$$ 

Note: By the previous observations, these contractions are equivariant with respect to the conjugation actions.

We note that this contraction extends to continuous maps:
\[ G : \mathbb{B}_n(e) \times \mathbb{R} \rightarrow S^3 \]
\[ (x,t) \rightarrow tx^t \]
\[ G : (S^3 \setminus \{-I\}) \times \mathbb{R} \rightarrow S^3 \]
\[ (A,t) \rightarrow A^t \] .

**Note:** \( G \) does not extend continuously over \( S^3 \times [0,1] \).

**Note:** The fact that \( \exp(i\theta) = e^{i\theta} \) follows from the previous observation concerning the metrics on \( S^3 \) and \( SU(2,\mathbb{C}) \). We shall frequently take advantage of these two perspectives (as well as others to be introduced below).

From Proposition 1.1 and the previous observations, we conclude that the extension of the contraction defined above is given by the following rule:

\[ G : (S^3 \setminus \{-I\}) \times \mathbb{R} \rightarrow S^3 \]
\[ (Ce^{i\theta}C^{-1},t) \rightarrow Ce^{i\theta}C^{-1} \]

where \( 0 \leq \theta < \pi \).

**Note:** The restriction on \( \theta \) is possible due to the fact that \( e^{i\theta} \) is conjugate to \( e^{-i\theta} \):

\[
\begin{pmatrix}
  e^{-i\theta} & 0 \\
  0 & e^{i\theta}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}
\begin{pmatrix}
  e^{i\theta} & 0 \\
  0 & e^{-i\theta}
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} .
\]

The next proposition describes the stabilizers of various elements in the "stratification" of \( S^3 \) above. For this reason, we use the standard notation:

\[ B \ast A = BAB^{-1} \]
\[ C \ast S^3 = \{CAC^{-1} \mid A \in S^3 \} \]
\[ S_A^3 = \{ B \in S^3 \mid B \ast A = A \} . \]

**Proposition 1.2:**

1. If \( A \in S^3 \), then \( S_A^3 = S^3 \).
2. If \( A \in (C \ast S^3) \setminus S^3 \), then \( S_A^3 = C \ast S^3 \).

**Proof:** (1) This is obvious.

(2) Suppose \( A \in (C \ast S^3) \setminus S^3 \). Since \( S^3 \) is an abelian group, it is clear that \( C \ast S^3 \subset S_A^3 \). We need to establish the reverse inclusion.

By part (1), \( C^{-1}A \in S^3 \). That is:

\[ C^{-1}AC = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} ; \ c^{i\theta} \neq e^{-i\theta} . \]

Suppose that \( B \in S_A^3 \), so that:

\[ BAB^{-1} = A \text{ or } (C^{-1}BC)(C^{-1}AC) = (C^{-1}AC)(C^{-1}BC) . \]

If \( C^{-1}BC = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) with \( aa + bb = 1 \), then:

\[
\begin{pmatrix}
  ac^{i\theta} & bc^{-i\theta} \\
  * & *
\end{pmatrix} =
\begin{pmatrix}
  ge^{i\theta} & be\theta \\
  * & *
\end{pmatrix} .
\]

Hence, \( b = 0 \), and \( B \in CS^1C^{-1} \) as before.
(d) The trace and argument maps

We have the usual trace function:

\[
\text{tr} : S^3 \rightarrow \mathbb{R} \\
\begin{pmatrix}
    a & b \\
    -\bar{b} & \bar{a}
\end{pmatrix} \mapsto a + \bar{a} .
\]

Of course, this function is \( SO_3 \) invariant:

\[
\text{tr}(CAC^{-1}) = \text{tr}(A) .
\]

Hence, by the results of the previous section, we have a surjection:

\[
\text{tr} : S^3 \rightarrow [-2,2] .
\]

Hence, we may define the argument function via the inverse cosine:

\[
\text{arg} : S^3 \rightarrow [0,\pi] \\
A \mapsto \cos^{-1}(\text{tr}(A)/2) .
\]

Then:

\[
\text{arg}(A) = 0 \iff A = \begin{pmatrix}
    e^{i\theta} & 0 \\
    0 & e^{-i\theta}
\end{pmatrix}
\]

\[
\text{arg}(A) = 0 \iff A = I
\]

\[
\text{arg}(A) = \pi \iff A = -I .
\]

The argument respects the natural contraction of the previous section:

\[
\text{arg}(A^t) = t \cdot \text{arg}(A) \quad A \neq -I \quad 0 \leq t \leq 1 .
\]
There is a natural involution of this decomposition:

\[ i : (S^7, D^7_+, D^7_-, D^7) \longrightarrow (S^7, D^7_-, D^7_+, D^7) \]

\[ A \longmapsto -A^{-1}. \]

We observe that this involution is $SO_3$ equivariant:

\[ i(CAC^{-1}) = Ci(A)C^{-1}. \]

In addition, it fixes $S^3$ pointwise:

\[ i(A) = A \text{ iff } A \in S^3. \]

Moreover, it is respected by the argument:

\[ \arg(-A^{-1}) = \pi - \arg(A). \]

(e) The unit quaternions $Sp(1)$ and $SO_3$.

We may identify $\mathbb{C}^2$ with the quaternions $\mathbb{H}$:

\[ \mathbb{C}^2 \longrightarrow \mathbb{H} \]

\[ (a, b) \longmapsto a + bj. \]

This identifies $S^3$ with the unit quaternions:

\[ S^3 = Sp(1) = \text{unit quaternions}. \]

Under this identification, the diffeomorphism of part (a) identifying $S^3$ with $SU(2, \mathbb{C})$ is an isomorphism of group structures and an isometry of metrics.

The action of $S^3$ on $\mathbb{H}$ by conjugation is a linear action by isometries of $S^3$. Hence, we obtain a representation:

\[ S^3 \longrightarrow SO_3 \]

\[ r(x)(y) = xyx^{-1} \]

Note: The representation is in $SO_4$ because $S^3$ is connected.

Now $R \subseteq \mathbb{H}$ and $R$ is fixed by the action. Hence, we can reduce $r$ to a representation in $SO_3$:

\[ S^3 \longrightarrow SO_3. \]

From Proposition 1.2, we conclude that the kernel of $r^i$ is $S^2$. Hence, $r^i$ is onto:

\[ 1 \longrightarrow S^0 \longrightarrow S^3 \longrightarrow SO_3 \longrightarrow 1 \text{ exact} \]

\[ SU(2, \mathbb{C}) = \frac{S^3}{S^2} = SO_3. \]

By Proposition 1.2, it follows that the conjugation action descends to a well-defined faithful action of $SO_3$ on $S^3$:

\[ \ast : SO_3 \times S^3 \longrightarrow S^3 \]

\[ ([A], B) \longmapsto ABA^{-1}. \]

Convention:

(1) We consider all of the above isomorphisms as identifications.

We observe that the natural involution of $S^3$ defined in the previous section, under the above identification with the unit quaternions, is given by the rule:
2. The space of representations of $G$, $R(G)$.

(a) the functor $R$

Let $G$ be a group. The space of representations of $G$ in $S^3$ is defined as follows:

$$R(G) = \text{Hom}(G, S^3)\,,$$

$$= \{ \rho : G \rightarrow S^3 \mid \rho \text{ is a representation} \} .$$

We give $R(G)$ the compact open topology, where $G$ has the discrete topology and $S^3$ the usual topology.

To each homomorphism $\lambda : G_1 \rightarrow G_2$, there is an associated continuous map:

$$R(\lambda) : R(G_1) \rightarrow R(G_2) \quad \rho \mapsto \rho \circ \lambda .$$

Clearly, we have the following identities:

$$R(\lambda \circ \mu) = R(\mu) \circ R(\lambda), \quad R(\text{Id}) = \text{Id} .$$

In other words, $R$ is a contravariant functor from the category of groups and homomorphisms to the category of topological spaces and continuous maps.

(b) The action of $SO_3$ on $R(G)$

By our previous remarks, the conjugation action of $S^3$ on $S^3$ descends to a faithful action of $SO_3$ on $S^3$:

$$* : SO_3 \times S^3 \rightarrow S^3 \quad \text{(faithful)} .$$

Given a group $G$, we may define an action of $SO_3$ on $R(G)$ (or an action of $SU_4(\mathbb{C})$):
This is a natural action in the following sense. For any homomorphism \( \lambda : G \to H \), the following diagram commutes:

\[
\begin{array}{ccc}
\text{SO}_3 \times R(H) & \xrightarrow{\text{id} \times R(\lambda)} & R(H) \\
\downarrow \text{SO}_3 \times R(\lambda) & & \downarrow R(\lambda) \\
\text{SO}_3 \times R(G) & \xrightarrow{\text{id} \times R(\lambda)} & R(G)
\end{array}
\]

Let:

\[
\begin{align*}
S_0(G) &= \{ \rho \mid \rho : G \to S^0 \} \\
S_1(G) &= \{ \rho \mid \rho : G \to S^1 \} = \text{diag. reps} \\
S(G) &= \{ \rho : G \to S^3 \text{ is reducible} \}.
\end{align*}
\]

**Proposition 2.1:**

1. \( S_0(G) \subset S_1(G) \subset S(G) \subset R(G) \)
2. \( S(G) = \cup \{ C \cdot S_1(G) \mid C \in SU_2(\mathbb{C}) \} \)

**Proof:**

1. This is immediate from the definitions.
2. Since the conjugate of a reducible representation is clearly reducible, part (1) implies that:

\[
S(G) \supset \cup \{ C \cdot S_1(G) \mid C \in SU_2(\mathbb{C}) \}.
\]

It remains to show that a reducible representation is conjugate to a diagonal representation. Suppose, therefore, that we have a reducible representation:

\[
\rho : G \to SU(2,\mathbb{C}) \text{ reducible.}
\]

By definition, \( \rho(G) \) leaves a proper subspace \( V \) of \( \mathbb{C}^2 \) invariant. The proof of Proposition 1.1 (2) yields an element \( C \in SU_2(\mathbb{C}) \) such that:

\[
C^{-1} \rho(g) C \in S^1 \text{ for all } g \in G.
\]

Equivalently, \( \rho \in C \cdot S_1 \).

The next corollary follows immediately from the definitions, Proposition 2.1 and the naturality of the \( \text{SO}_3 \) action.

**Corollary 2.2:** If \( \lambda : G \to H \) is a homomorphism, then we have a commutative diagram:

\[
\begin{array}{ccc}
S_0(H) & \subset & S_1(H) \\
\downarrow R(\lambda) & & \downarrow R(\lambda) \\
S_0(G) & \subset & S_1(G) \\
\end{array}
\]

**Note:** In fact, the decomposition of Proposition 1.1(2) is also natural.

The next proposition describes the stabilizers of representations in this "stratification" of \( R(G) \). For that purpose, we use the standard notation:

\[
S^3_\rho = \{ A \in S^3 \mid A \cdot \rho = \rho \}.
\]

Clearly:

\[
A \in S^3_\rho \text{ iff } A \in S^3_{\rho(g)} \text{ for all } g \in G.
\]

\[
A \in S^3_\rho \text{ iff } \rho(g) \in S^3_A \text{ for all } g \in G.
\]
Proposition 2.3:

1. If \( \rho \in S_\rho(G) \), then \( S^\rho_\rho = S^3 \).
2. If \( \rho \in \mathcal{C} \cdot S_1(G) \setminus S\rho(\rho) \), then \( S^\rho_\rho = CS^1 _\rho c^{-1} \).
3. If \( \rho \in R(G) \setminus S(G) \), then \( S^\rho_\rho = S^0 \).

Let:

\[ \overline{R}(G) = S\rho \setminus \overline{R}(S) \]

= space of conjugacy classes of reps.

of \( G \) in \( S^3 \).

By the previous observations, we conclude that if \( \lambda : G \rightarrow H \) is a homomorphism, there is a continuous map:

\[ \overline{R}(\lambda) : \overline{R}(H) \longrightarrow \overline{R}(G) \]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\overline{R}(H) & \xrightarrow{\overline{R}(\lambda)} & \overline{R}(G) \\
\downarrow{\pi} & & \downarrow{\pi} \\
\overline{R}(H) & \xrightarrow{\overline{R}(\lambda)} & \overline{R}(G) \\
\end{array}
\]

Let:

\[ \overline{S}_\chi(G) = \pi(S_\chi(G)) \]

\[ \overline{S}_\chi(G) = \pi(S_\chi(G)) \]

\[ \overline{E}(G) = \pi(S(G)) = S\rho \setminus S(G) \]

The diagram below commutes:

\[
\begin{array}{ccc}
\overline{S}_\chi(H) & \xrightarrow{\overline{S}_\chi(\lambda)} & \overline{S}(H) \\
\downarrow{\overline{R}(\lambda)} & & \downarrow{\overline{R}(\lambda)} \\
\overline{S}_\chi(G) & \xrightarrow{\overline{S}_\chi(\lambda)} & \overline{S}(G) \\
\end{array}
\]

(c) The algebraic set associated to a set of generators, \( S_\chi \), of \( G \)

Suppose that \( G \) is a finitely generated group. Consider a finite set of generators for \( G \):

\[ S = \{ a_1, \ldots, a_n \} \]

We have a natural map:

\[ f = f_S : R(G) \longrightarrow (S^3)^n \]

\[ \rho \longmapsto (\rho(a_1), \ldots, \rho(a_n)) \]

Let:

\[ V(S) = f_S(\overline{R}(G)) \]

Now, if \( R \) is a set of relations for \( G \), so that:

\[ G = \langle a_1, \ldots, a_n \mid r(a_1, \ldots, a_n) = 1 \rangle \]

is a presentation of \( G \), then:

\[ V(S) = \{(A_1, \ldots, A_n) \in (S^3)^n \mid r(A_1, \ldots, A_n) = 1 \} \]

Hence, we have a natural homeomorphism from \( R(G) \) to a real algebraic set.

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\( f_S : R(G) \rightarrow V(S) \)
\[ \rho \mapsto (\rho(g_1), \ldots, \rho(g_n)) . \]

Note: \( V(S) \) is actually the locus of finitely many polynomial equations:
\[ r_j(a_1, \ldots, a_n) = 1 \quad r_j \in R \quad 1 \leq j \leq m . \]

Hence, \( R(G) = R(H) \), where \( H \) is a finitely presented group with quotient group \( G \).

(d) Induced polynomials maps

If, in addition, we are given another group \( H \), a finite generating set for \( H \), and a homomorphism \( \lambda : G \rightarrow H \), then there is a unique polynomial map \( \lambda^T_S \) completing the following commutative diagram:

\[
\begin{array}{ccc}
R(H) & \xrightarrow{R(\lambda)} & R(G) \\
\downarrow & & \downarrow \\
V(T) & \xrightarrow{\lambda^T_S} & V(S)
\end{array}
\]

Let:
\[ T = \{ h_1, \ldots, h_n \} . \]

If we express the images of generators of \( G \) as words in the generators of \( H \):
\[ \lambda(g_j) = h_{s_1}^{a_1} \ldots h_{s_p}^{a_p} \] 1 \( \leq j \leq n \]
then \( \lambda^T_S \) is given by the rule:

\[ \lambda^T_S(A_1, \ldots, A_m) = (A_{s_1}^{a_1}, \ldots, A_{s_p}^{a_p}) . \]

Note: We may assume, as above, that \( p \) is independent of \( j \) by introducing trivial exponents if necessary.

These induced polynomial maps behave very much like matrices in the theory of linear transformations. In particular, if \( H \) is a third group with finite generating set \( U \) and \( \gamma \) is a homomorphism from \( H \) to \( K \), then:
\[ (\gamma \circ \lambda)^U_S = \lambda^T_S \cdot \gamma^T_T . \]

Likewise:
\[ (\text{id})^S_S = \text{id} . \]

(e) The real algebraic structure of \( R(G) \)

From these observations, we observe that if \( \lambda : G \rightarrow H \) is an isomorphism, then the induced polynomial map is an algebraic isomorphism of real algebraic sets:
\[ \lambda^T_S : V(T) \xrightarrow{\sim} V(S) . \]

It follows that \( R(G) \) has a well defined structure of a real algebraic set induced from the natural homeomorphisms associated to finite generating sets of \( G \).

Conventions: Given a group \( G \) with a finite generating set \( S \), we make the identification:
\[ R(G) = V(S) . \]
Moreover, given a homomorphism \( \lambda : G \rightarrow \Pi \) and finite generating set \( T \) for \( \Pi \):

\[
R(\lambda) = \lambda_S^T.
\]

When we do not need to emphasize the generating sets we shall write:

\[
\lambda^* = \lambda_S^T.
\]

\([\text{eg}]\)

\[
G = \langle x \mid x \in \mathbb{Z} \rangle \quad H = \langle y \mid y^2 = 1 \rangle \cong \mathbb{Z}_2
\]

\[
\lambda : G \rightarrow H \quad S = \{x\} \quad T = \{y\}
\]

\[
\lambda^* : V(T) \rightarrow V(S) \quad \text{inclusion}
\]

\[
\ast 1 \rightarrow \ast 1
\]

3. **Representations of a free group.**

   (a) **Standard model with respect to a basis**

   Let:

   \[
   G = \langle g_1, \ldots, g_n \rangle = \text{free group of rank } n.
   \]

   Then:

   \[
   R(G) \xrightarrow{\rho} (S^3)^n
   \]

   \[
   \rho \mapsto (\rho(g_1), \ldots, \rho(g_n)).
   \]

   If \( \lambda : G \rightarrow G \) is an endomorphism, then:

\[
R(\lambda) : (S^3)^n \rightarrow (S^3)^n
\]

\[
(\rho(g_1), \ldots, \rho(g_n)) \mapsto (\rho(\lambda(g_1)), \ldots, \rho(\lambda(g_n))).
\]

**Note:** As in the previous example, if \( \Pi \) is a quotient of \( G \), then the quotient map:

\[
\lambda : G \rightarrow \Pi
\]

induces an inclusion of representation spaces:

\[
R(\lambda) : R(H) \rightarrow R(\Pi).
\]

This is a general property of Hom functors.

(b) **The homology of** \( R(G) \).

Since \( R(G) \) is 2-connected, we have the natural isomorphism:

\[
\alpha : H^2(R(G), \mathbb{Z}) \xrightarrow{\cong} \text{Hom}(H_5(R(G), \mathbb{Z}), \mathbb{Z}).
\]

In addition, we have the commutative diagram:

\[
\begin{array}{ccc}
H^2(R(G), \mathbb{Z}) & \xrightarrow{\rho^*} & H^2(R(G), \mathbb{Z}) \\
\text{Hom}(H_5(R(G), \mathbb{Z}), \mathbb{Z}) & \downarrow{\cong} & \text{Hom}(H_5(R(G), \mathbb{Z}), \mathbb{Z}) \\
\text{Hom}(H_5(R(G), \mathbb{Z}), \mathbb{Z}) & \downarrow{\cong} & \text{Hom}(H_5(R(G), \mathbb{Z}), \mathbb{Z})
\end{array}
\]

For each element \( g \) of \( G \), we have a natural evaluation map:
\[
\begin{align*}
\text{ev}_g : R(G) & \longrightarrow S^3 \\
\rho & \longrightarrow \rho(g)
\end{align*}
\]

We have the following identity:
\[
\text{ev}_{gh}(\rho) = \text{ev}_g(\rho) \cdot \text{ev}_h(\rho).
\]
Likewise, since \(S^3\) is 2-connected, we have the natural isomorphism:
\[
\beta : H^2(S^3, \mathbb{Z}) \longrightarrow \text{Hom}(H_2(S^3), \mathbb{Z}).
\]
Let \(\mu\) be the "dual" to the top class \(z\) of \(S^3\). (Recall that we fixed an orientation on \(S^3\).)
\[
\begin{align*}
\mu : H_2(S^3, \mathbb{Z}) & \longrightarrow \mathbb{Z} \\
z & \longmapsto 1
\end{align*}
\]
\(z\) = fund. class of \(S^3\).

Using the above observations, we obtain a natural homomorphism:
\[
\begin{align*}
\varepsilon^* : H_1(G, \mathbb{Z}) & \longrightarrow H^2(R(G), \mathbb{Z}) \\
g & \longmapsto \text{ev}_g^*(\mu)
\end{align*}
\]
If \(\lambda\) is an endomorphism of \(G\), then we have the following identity:
\[
\text{ev}_g \cdot R(\lambda) = \text{ev}_{\lambda(g)}.
\]
The next proposition follows immediately.

**Proposition 3.1:** Suppose \(\lambda\) is an endomorphism of \(G\). Then the following diagram commutes:

\[
\begin{align*}
H_1(G, \mathbb{Z}) & \longrightarrow R_1(G, \mathbb{Z}) \\
\varepsilon^* & \downarrow \quad \lambda^* \downarrow \\
H^2(R(G), \mathbb{Z}) & \longrightarrow H^2(R(G), \mathbb{Z}).
\end{align*}
\]

We consider the naturally oriented 3-cycles in \(R(G)\):
\[
\begin{align*}
x_j : S^3 & \longrightarrow R(G) \\
A & \longmapsto (A_{ij}, \ldots, A_{kj})
\end{align*}
\]
where \(A_{ij} = I\) if \(i \neq j\)
\(= A\) if \(i = j\).

Since \(G\) is a free group, we obtain an homomorphism:
\[
\varepsilon : G \longrightarrow \pi_3(R(G))
\]
\(g \longmapsto x_j\).

**Note:** Given the identification of \(R(G)\) with \((S^3)^{\mathbb{N}}\), \(R(G)\) is a topological group. Hence, the homomorphism \(\varepsilon\) is given by taking products with respect to the group structure of \(R(G)\). This agrees with the usual structure on \(\pi_3\).

Since \(\pi_3(R(G))\) is an abelian group, \(\varepsilon\) factors through \(G/[G,G]\):
\[
\varepsilon : H_1(G, \mathbb{Z}) \longrightarrow \pi_3(R(G))
\]
\(g \longmapsto x_j\).

Finally, we can compose with the Hurewicz homomorphism:
$\varepsilon : H_1(G, \mathbb{Z}) \xrightarrow{\varphi} H_3(R(G), \mathbb{Z})$

$\delta_j \longrightarrow x_j$

This last map is an isomorphism, since \(\{\delta_1, \ldots, \delta_n\}\) is a basis for \(H_1(G, \mathbb{Z})\) and \(\{x_1, \ldots, x_n\}\) is, clearly, the standard basis for \(H_3((S^3)^n)\).

\[ H_1(G, \mathbb{Z}) = \langle \delta_1, \ldots, \delta_n \rangle \cong \mathbb{Z}^n \]

\[ H_3(R(G), \mathbb{Z}) = \langle x_1, \ldots, x_n \rangle \cong \mathbb{Z}^n \]

Now, the Hurewicz homomorphism yields an isomorphism:

\[ \gamma : H_3(S^3) \xrightarrow{\cong} H_3(S^3, \mathbb{Z}) \].

Under this identification, the top class \(z\) of \(S^3\) is given by the identity map:

\[ z = [\text{id}] \quad \text{id} : S^3 \longrightarrow S^3 \]

\[ A \longrightarrow A \]

We compute that \(\{\varepsilon^*(\delta_1), \ldots, \varepsilon^*(\delta_n)\}\) is a dual basis for \(H^3(R(G), \mathbb{Z})\):

\[ \varepsilon^*(\delta_i)(x_j) = \delta_i^j = 1 \quad \text{if} \quad i = j \\
= 0 \quad \text{if} \quad i \neq j \]

Hence, \(\varepsilon^*\) is a natural isomorphism:

\[ \varepsilon^* : H_1(G, \mathbb{Z}) \xrightarrow{\cong} H^3(R(G), \mathbb{Z}) \]

\[ \delta_j \longrightarrow x_j^* \]

The next proposition follows immediately.

**Proposition 3.2:** Suppose \(\lambda\) is an endomorphism of \(G\). Then the following diagram commutes:

\[ \xymatrix{ H_1(G, \mathbb{Z}) \ar[r]^{\lambda_\ast} \ar[d]_{\varepsilon} & H_1(R(G), \mathbb{Z}) \ar[d]^{\pi} \\
H_3(R(G), \mathbb{Z}) \ar[r]^{(\lambda_\ast)^t} & H_3(R(G), \mathbb{Z}) } \]

where \(\lambda_\ast\) is an induced homomorphism of \(H_1(G, \mathbb{Z})\)

\[ \lambda_\ast = (\rho(\lambda))_{\ast} = \text{ind. hom. of} \ H_3(R(G), \mathbb{Z}) \]

and \(t = \text{transpose w.r.t. the ordered basis} \ (x_1, \ldots, x_n) \).

(c) **The tangent bundle of** \(R(G)\)

We fix a standard model for \(R(G)\) as in section (a):

\[ G = \langle \delta_1, \ldots, \delta_n \rangle \]

\[ R(G) = (S^3)^n \quad \rho = (\lambda_1, \ldots, \lambda_n) \]

We see that:

\[ T_R(G) = \{(A_1, \ldots, A_n, x_1, \ldots, x_n) \mid A_i \in S^3, x_i \in T_{A_i}(S^3)\} \]

Let:

\[ 1 = (I, \ldots, I) = \text{trivial representation} \]

Then the "Lie Algebra" of \(R(G)\) is given as:

\[ T_1(R(G)) = S^3 = S_{1} \oplus \cdots \oplus S_n \]

where \(S\) is the Lie algebra of \(S^3\) described in section 1(b). As for \(S^3\), we have a "natural" trivialization:
\[ E(G) \times \mathbb{R} \xrightarrow{\mathbb{R}} T(R(G)) \]
\[
(A_1, \ldots, A_n, X_1, \ldots, X_n) \mapsto (A_1 A_1^{-1}, \ldots, A_n A_n^{-1})
\]

Likewise, we have a "natural" Riemannian metric arising via translation from the inner product on \( \mathbb{R}^n \):

\[
\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}
\]
\[
((X_1, \ldots, X_n), (Y_1, \ldots, Y_n)) \mapsto \sum_i \text{tr}(X_i Y_i^t)
\]

With respect to these models, the conjugation actions of \( SO_3 \) (or \( \mathbb{S}^3 \)) are given by:

\[
\cdot : SO_3 \times R(G) \rightarrow R(G)
\]
\[
([A], (A_1, \ldots, A_n)) \mapsto (AA_1 A_1^{-1}, \ldots, AA_n A_n^{-1})
\]
\[
\cdot : SO_3 \times T(R(G)) \rightarrow T(R(G))
\]
\[
([A], (A_1, \ldots, A_n)) \mapsto ((AA_1 A_1^{-1})_1, (AA_2 A_2^{-1})_2, \ldots, (AA_n A_n^{-1})_n)
\]

Again, these actions are actions by isometries.

**Note:** We are inclined to consider \( R(G) \) as a Lie group via the identification with \((\mathbb{S}^3)^n\). Although this is possible, it is not natural; the induced group structure depends strongly on the choice of basis. Nevertheless, this auxiliary structure is useful for technical purposes.

We now wish to compare the action of \( \text{End}(G) \) on \( H_1(G, \mathbb{Z}) \) with the action on the "Lie Algebra" of \( R(G) \), \( T_1(R(G)) \). As in Proposition 3.1, we construct an identification.

Note: We may also consider cotangent spaces. This is most natural, as we saw in Proposition 2.3. We shall not develop this approach.

Using the given basis for \( H_1(G, \mathbb{Z}) \) we can identify \( T_1(R(G)) \) with \( H_1(G, \mathbb{Z}) \): \( H_1(G, \mathbb{Z}) = H_1(G, \mathbb{Z}) \oplus \mathbb{S} \)
\[
= \mathbb{S} \oplus \mathbb{S} \oplus \cdots \oplus \mathbb{S} = T_1(R(G))
\]

We obtain the following analog of Proposition 2.3:

**Proposition 3.3:** Suppose \( \lambda \) is an endomorphism of \( G \). Then the following diagram commutes:

\[
\begin{array}{ccc}
H_1(G, \mathbb{Z}) \oplus \mathbb{S} & \xrightarrow{\lambda_* \oplus \text{id}} & H_1(G, \mathbb{Z}) \oplus \mathbb{S} \\
\downarrow & & \downarrow \\
T_1(R(G)) & \xrightarrow{D_1 R(\lambda)} & T_1(R(G))
\end{array}
\]

**Proof:** Observe that the tangent space at \( 1 \) to \( R(G) \) is spanned by curves of cyclic representations, curves lying in the 3-cycles in the proof of Proposition 3.2. A typical curve has the form:

\[
\begin{align*}
\omega_j(t) &= (A_{ij}(t), \ldots, A_{ij}(t)) \\
A_{ij}(t) &= I \quad \text{if } i \neq j \\
A_{jj}(0) &= I
\end{align*}
\]

Write the images of generators of \( G \) under \( \lambda \) as words in the generators of \( G \):

\[
\lambda(g) = \omega_j(g_1, \ldots, g_n)
\]
Then the action of $\lambda$ on $E_i(G, Z)$ is given by the following:

\[(i) \quad \lambda^\ast \omega \left( g_j \right) = \sum_{i=1}^{n} \exp_S \left( \omega_{i} \right) g_i \]

where $\exp_S \left( \omega_{i} \right)$ is the sum of the exponents of occurrences of $g_i$ in $w_j$.

On the other hand, we compute the action on the tangent space as follows:

$$R(\lambda) (e_i(t)) = R(\lambda) (A(t), 1, \ldots, 1) = ((A(t))^{\exp_{g_i} (\omega_i)}, \ldots, (A(t))^{\exp_{g_i} (\omega_n)})$$

\[\left[ \frac{d (R(\lambda) (e_i(t)))}{dt} \right]_{t=0} = \left( \exp_{g_i} (\omega_i) \frac{\partial A}{\partial t} \bigg|_{t=0}, \ldots, \exp_{g_i} (\omega_n) \frac{\partial A}{\partial t} \bigg|_{t=0} \right).\]

Under the identification of $T_1(R(G))$ with $H_1(G, Z) \otimes S$ described above, the tangent vector to the curve $e_i(t)$ at the trivial representation is clearly $g_i \otimes x$, where

$$X = \left( \frac{\partial A}{\partial t} \bigg|_{t=0} \right) \in S.$$

Generalizing, we conclude that:

\[(ii) \quad D(\lambda) \omega \left( x_j \otimes X \right) = \sum_{i=1}^{n} \exp_{g_i} (\omega_i) x_i \otimes X \]

Comparing (i) and (ii), we are done. $\square$

(d) Induced orientations of $\mathbb{R}(G)$ via a basis of $G$

Given a basis for $G$, we use the identification of part (a) to induce an orientation on $\mathbb{R}(G)$. The fixed orientation of $S^1$ induces the obvious orientation on $(S^n)^n$. That is, by the Kunneth formulas:

$$H_i(S^n) \otimes \cdots \otimes H_i(S^n) \quad \longrightarrow \quad H_{n^i}(S^n)^n$$

and the fundamental class $w$ is the image:

$$z \otimes \cdots \otimes z \quad \longrightarrow \quad w = z \otimes \cdots \otimes z .$$

In a similar manner, as we have seen, we obtain an identification:

$$T_i(S^n) \otimes \cdots \otimes T_i(S^n) \quad \longrightarrow \quad T_i(R(G))$$

or more generally:

$$T_{A_1}(S^n) \otimes \cdots \otimes T_{A_n}(S^n) \quad \longrightarrow \quad T_{A}(R(G))$$

where $(A_1, \ldots, A_n) = (\rho(g_1), \ldots, \rho(g_n))$.

If we denote by $\tilde{z}$ and $\tilde{w}$ the orientations on the tangent spaces induced by $z$ and $w$, we see that:

$$\tilde{z} \otimes \cdots \otimes \tilde{z} \quad \longrightarrow \quad \tilde{w} .$$

We consider the following bilinear forms. First, we have the standard pairing of homology and cohomology:

$$\langle , \rangle : H^p(R(G), Z) \times H_p(R(G), Z) \longrightarrow Z .$$

Secondly, we have the cup product:

$$\cup : H^p(R(G), Z) \times H^q(R(G), Z) \longrightarrow H^{p+q}(R(G), Z) .$$

Likewise, we have the cap product:

$$\cap : H^p(R(G), Z) \times H_q(R(G), Z) \longrightarrow H_{p-q}(R(G), Z) .$$
recall that there is a natural identification:

\[ H_p(R(G), \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z} \, . \]

Under this identification, we have:

\[ \langle \alpha, \beta \rangle = n \text{ in case } p = q \, . \]

Furthermore, \( u \) and \( n \) are dual:

\[ \langle \alpha u \beta, y \rangle = \langle \alpha, \beta \rangle n y \, . \]

Given an orientation of \( R(G) \) as above, we obtain the Poincaré duality:

\[ D_w : H^p(R(G), \mathbb{Z}) \xrightarrow{\sim} H_{n-p}(R(G), \mathbb{Z}) \, . \]

\[ \alpha \longmapsto \alpha \cap w \, . \]

Furthermore, by "counting" signed intersections with respect to this orientation, we obtain the intersection product:

\[ \langle \cdot, \cdot \rangle_w : H_p^w(R(G), \mathbb{Z}) \times H_q^w(R(G), \mathbb{Z}) \longrightarrow H_{p+q-w}(R(G), \mathbb{Z}) \, . \]

Again, by using the identification above, we obtain as a special case:

\[ \langle \cdot, \cdot \rangle_w : H_p^w(R(G), \mathbb{Z}) \times H_q^w(R(G), \mathbb{Z}) \longrightarrow \mathbb{Z} \, . \]

The fundamental identity relating these various forms is:

\[ \langle D(\alpha), D(\beta) \rangle_w = D(\alpha \cap \beta) \, . \]

In particular, in the special case, duality arises via the formula:

\[ \langle D(\alpha), D(\beta) \rangle_w = \langle \alpha, \beta \rangle, \quad \dim(\alpha) + \dim(\beta) = 3n \, . \]

Considering the natural map \( \varepsilon^* \) of part (b), we easily compute that:

\[ \langle \varepsilon^*(g_1), \ldots, \varepsilon^*(g_n), w \rangle = \prod_{i=1}^n \varepsilon^*(g_i)(x_i) = 1 \, . \]

Hence, the "dual" of \( w \) is given by:

\[ w^* = x_1^* \cup \ldots \cup x_n^* \quad x_i^* = \varepsilon^*(g_i) \quad 1 \leq i \leq n \, . \]

Note: As in section 3(b), "dual" refers to the dual in the functional sense. We reserve the term dual for Poincaré dual.

From Propositions 3.2 and 3.3, and these observations, we deduce the following proposition.

**Proposition 3.4:** If \( \lambda \) is an endomorphism of \( G \), then

1. \( R(\lambda)^*(w^*) = \det(\lambda_1)w^* \, . \)
2. \( R(\lambda_1)(w) = \det(\lambda_1)w \, . \)

Suppose \( g \) is a generator of \( G \) (an element of a basis of \( G \)).

Let:

\[ y(g) = \{ \rho \mid \rho(g) = 1 \} \, . \]

Choosing a basis for \( g \) which contains \( g_i \), we can order the basis so that:

\[ g = \langle g_1, \ldots, g_n \rangle \quad g = \rho(g_1) \, . \]

With respect to the associated identification, we obtain an homomorphism:

\[ y(g) \xrightarrow{\cong} (\mathbb{Z})^{n-1} \]

\[ \rho(g_1) \longmapsto (\rho(g_1), \ldots, \rho(g_n)) \, . \]

Hence, \( y(g) \) determines a \( 2n - 3 \) cycle in \( R(G) \) which is well defined up to sign.
CHAPTER II: HEGGARD DECOMPOSITIONS AND STABLE EQUIVALENCE

1. Heggard decompositions and models

(a) The standard handlebody \( W \)

Let:

\[
W = \text{standard (model) handlebody of genus } g \ (g \geq 1) \\
F = \partial W = \text{boundary of } W \\
D = \text{embedded 2-disk in } F \\
0 = \text{basepoint of } F \text{ on } \partial D \\
F^* = F \setminus \text{interior of } D, \quad S^1 = \partial D .
\]

![Figure 4](image)

We may choose a family of loops on \((F^*, 0)\), as in Figure 4, such that:
Let $c_i$ and $d_i$ denote the homology classes of $a_i$ and $b_i$ respectively. Then we have abelian presentations:

\[ H_1(F, \mathbb{Z}) = H_1(F^g, \mathbb{Z}) = \langle c_1, d_1, \ldots, c_g, d_g \rangle \cong \mathbb{Z}^g \]

\[ H_1(W, \mathbb{Z}) = \langle c_1, \ldots, c_g \rangle \cong \mathbb{Z}^g. \]

(b) The Heegaard model \((W, h)\)

Given an orientation preserving homeomorphism:

\[ h : (F, D, 0) \xrightarrow{\cong} (-F, -D, 0) \text{ or. pres.} \]

where \((-F, -D)\) is \((F, D)\) with the opposite orientation, we define a closed orientable 3-manifold:

\[ |(W, h)| = \{(W \times [1]) \cup (-W \times [2]) \} \subset \langle \langle x(1) - (h(x), 2) \mid x \in F \rangle \} \]

(c) Heegaard decomposition of a three-manifold \(M^3\)

An Heegaard decomposition of a three-manifold \(M^3\) is a pair of subsets \((W_1, W_2)\) of \(M^3\) with the following properties:

\[ W_1 \cup W_2 = M^3 \]

\[ W_1 \cap W_2 = \emptyset. \]

Given an Heegaard decomposition of \(M^3\), we may construct an homeomorphism to an Heegaard model as follows:

(1) \[ W_i = W_1 \cup W_2 \]

(2) \[ \partial W_1 = W_1 \cap W_2 = \partial W_2 \]

(3) \[ W_1 \cup W_2 = M^3. \]

Note: We choose \(f_2\) and \(f_1\) so that \(h\) is an automorphism of the triple \((F, D, 0)\) as above.

(6) \[ f : M^3 \xrightarrow{h} |(W, h)| \]

\[ x \xrightarrow{h} ([f_1(x), 1]) \quad x \in W_1 \]

\[ x \xrightarrow{h} ([f_2(x), 2]) \quad x \in W_2. \]

It is well known that every closed 3-manifold has an Heegaard decomposition. Hence, every closed 3-manifold is homeomorphic to an Heegaard model. We shall work with Heegaard models for the most part. This is primarily done for purely computational and technical reasons.

(d) Associated presentations of \(\pi_1\) and \(H_1\)

Identify \(M^3\) with an Heegaard model \((W, h)\). By the Van Kampen Theorem, we may obtain a presentation for \(H_1(M^3, C)\) as follows:
Alternatively, we can obtain a presentation by attaching discs to $F$ and capping off holes with 3 discs. We obtain $M^g$ by attaching discs to $b_1, \ldots, b_g$ and capping off the resulting 2-sphere to get $W_1$, and then attaching discs to $h^{-1}(b_1), \ldots, h^{-1}(b_g)$ and capping to attach $W_2$. This yields the following presentation:

$$
\Pi_1(M^g, 0) = \left\langle \frac{a_1^* \cdot a_2^* \cdot \ldots \cdot a_g^*}{b_1} \left| \begin{array}{l}
(1) \quad b_1 = 1 \\
(2) \quad b_1^{-1}(b_1) = 1
\end{array} \right. \right. 
$$

Here we have identified $(W, 0)$ with $W \times \{1\} = \{W, h\}$ via the map $j$ and $(-W, 0)$ with $-W \times \{2\}$ via $k$.

Then we conclude that:

$$
\Pi_1(M^g, 0) = \left\langle \frac{a_1^* \cdot a_2^* \cdot \ldots \cdot a_g^*}{b_1} \left| \begin{array}{l}
(1) \quad i(a_1) = (i.h)_g(a_1) \\
(2) \quad i(b_1) = (i.h)_g(b_1)
\end{array} \right. \right. 
$$

These two presentations, of course, are of the same type. The first considers $F$ as identified with the target of $h$, the latter with the source. We shall prefer the latter perspective.

Abelianizing, we obtain $H_1(M^g, \mathbb{Z})$:

$$
\Pi_1(M^g, 0) = \left\langle \frac{a_1^* \cdot a_2^* \cdot \ldots \cdot a_g^*}{(i.h)_g(b_1)} \left| \begin{array}{l}
(1) \quad d_1 = 1 \\
(2) \quad h_g^{-1}(d_1) = 1
\end{array} \right. \right. 
$$
Since \( F \) is oriented, we have a well defined intersection pairing on \( H_1(F;\mathbb{Z}) \):
\[
\langle \cdot, \cdot \rangle : H_1(F,\mathbb{Z}) \times H_1(F,\mathbb{Z}) \to \mathbb{Z}
\]
We assume that the basis for \( H_1(F,\mathbb{Z}) \) described above is chosen so that this pairing has the following matrix with respect to the ordered basis \( \{c_1, \ldots, c_g, d_1, \ldots, d_g\} \):
\[
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad (c_1, d_1) = 1.
\]
That is, if we consider the pairing:
\[
\langle \cdot, \cdot \rangle : H_1(F,\mathbb{Z}) \times H_1(F,\mathbb{Z}) \to \mathbb{Z}
\]

(1) \( \langle c_i, c_j \rangle = \delta_{i,j} \)

(2) \( \langle c_i, d_j \rangle = \langle d_j, c_i \rangle = 0 \)

(3) \( \langle d_i, d_j \rangle = \delta_{i,j} \)

we have the identity:
\[
(x, y)_P = \langle x, y \rangle.
\]
Recall that \( h \) is an orientation preserving homeomorphism from \( F \) to \(-F\).
It follows immediately from the definition of the intersection product that:
\[
\langle h_\ast(x), h_\ast(y) \rangle_p = -(x, y)_p.
\]
Combining these identities, we compute:
\[
\langle x, y \rangle = \langle x, y \rangle_p = -(h_\ast(x), h_\ast(y))_p
\]
\[
\langle x, y \rangle = -\langle h_\ast(x), Jh_\ast(y) \rangle
\]
\[
\langle x, y \rangle = \langle x, -h_\ast^{-1}Jh_\ast y \rangle.
\]
Hence:
\[
J = -h_\ast^{-1}Jh_\ast
\]
\[
h_\ast^{-1} = -J^{-1}h_\ast^t \] where \( J^{-1} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \).
Writing \( h_\ast \) and \( h_\ast^{-1} \) with respect to this basis, we obtain the equivalence:
\[
h_\ast = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Rightarrow h_\ast^{-1} = \begin{bmatrix} B^t & -A^t \\ -C^t & A^t \end{bmatrix}.
\]
Hence, we obtain a presentation matrix for \( H_1(M^3,\mathbb{Z}) \) \( P \):
\[
P = \begin{bmatrix} 0 & A^t \\ I & -A^t \end{bmatrix} = (b_1, \ldots, b_g, h_\ast^{-1}(b_1), \ldots, h_\ast^{-1}(b_g)).
\]
That is to say, we have a short exact sequence:
\[
1 \to \mathbb{Z}^2 \overset{P}{\to} \mathbb{Z}^2 \overset{P}{\to} H_1(M^3,\mathbb{Z}) \to 1.
\]
In particular, we conclude that:
\[
\rho_1(M^3) \neq 0 \iff \det(P) = 0
\]
\[
\rho_1(M^3) = 0 \iff \det(P) = |\Pi_1(M^3,\mathbb{Z})|
\]
where \( |\Pi_1(M^3,\mathbb{Z})| \) is the order of \( \Pi_1(M^3,\mathbb{Z}) \).
2. Stable equivalence of Heegard decompositions

(a) Stable equivalence of Heegard decompositions

Suppose $M^3$ is a closed 3-manifold with Heegard decompositions $(W_1, W_2)$ and $(W_1', W_2')$. We say that $(W_1, W_2)$ is isotopic to $(W_1', W_2')$ if there is an ambient isotopy of $M^3$ taking $W_1$ to $W_1'$. We write $(W_1, W_2) \approx (W_1', W_2')$.

An unknotted one handle $H$ in an handlebody $W$ is an embedding of $D^2 \times I$ such that:

1. $H \cap F = D^2 \times \{0,1\}$ ($F = \partial W$)
2. if $a$ is the arc $[0,1] \subset H$ there is an arc $\beta$ on $F$ such that $a \cup \beta$ is the boundary of an embedded disc $B^2$ with $B^2 \cap F = \beta$.

Note: Up to ambient isotopy of $W$ there is exactly one unknotted one handle $H$ in $W$.

Given an unknotted one handle $H$ in $W_1$, we obtain a new Heegard decomposition of $M^3$:

$$
\sigma(W_1, W_2) = (W_1 \cup H, W_2 \setminus H)
$$

well defined up to isotopy.

![Figure 6](image)

![Figure 7](image)

Figure 7

We say that $(W_1, W_2)$ is stably equivalent to $(W_1', W_2')$ if there exists a pair of nonnegative integers, $j, k \geq 0$, such that

$$
\sigma^j(W_1, W_2) = \sigma^k(W_1', W_2')
$$

Note: This move of adding an unknotted one handle to increase the genus of an Heegard decomposition has an alternative interpretation in terms of connected sums of Heegard decompositions. (See sections (c) and (d) below).

(b) Singer's Theorem on Stable Equivalence of Heegard Decompositions of Closed Orientable Three Manifolds (SIM)

Stable equivalence is the basic equivalence relating the Heegard decompositions of a fixed three-manifold. This is the theorem of Singer:
Theorem 2.1 (Singer): Any two Heegaard decompositions of an orientable closed 3 manifold are stably equivalent.

(c) Connected sums of Heegaard decompositions

Suppose that \((W_1,W_2)\) is a Heegaard decomposition of \(M^3\) and \((V_1,V_2)\) is an Heegaard decomposition of \(N^3\). Choose a 3-ball \(B\) in \(M^3\) and a 3-ball \(C\) in \(N^3\) such that:

- \(B \cap \partial W_1 = B = 2\text{-disc}\)
- \(B \cap \partial B = \partial B\)
- \(C \cap \partial V_1 = C = 2\text{-disc}\)
- \(C \cap \partial C = \partial C\)

\[\begin{tikzpicture}
    \node (W1) at (0,0) {\(W_1\)};
    \node (V1) at (2,0) {\(V_1\)};
    \draw[fill] (0,0) circle (0.3) node [below] {\(B\)};
    \draw[fill] (2,0) circle (0.3) node [below] {\(C\)};
    \draw[thick,->] (0,0) -- (2,0);
    \end{tikzpicture}\]

Figure 8

If we remove the interiors of \(B\) and \(C\) from \(M^3\) and \(N^3\) respectively and glue \(\partial B\) to \(\partial C\) by a homeomorphism respecting the decomposition:

\[\varphi: (\partial B, \partial B \cap W_1) \rightarrow (\partial C, \partial C \cap V_1)\]

we evidently obtain an Heegaard decomposition for the connected sum of \(M^3\) and \(N^3\):

\[\begin{tikzpicture}
    \node (W1) at (0,0) {\(W_1\)};
    \node (V1) at (2,0) {\(V_1\)};
    \draw[fill] (0,0) circle (0.3) node [below] {\(B\)};
    \draw[fill] (2,0) circle (0.3) node [below] {\(C\)};
    \draw[thick,->] (0,0) -- (2,0);
    \end{tikzpicture}\]

\[\begin{align*}
(W_1,W_2) \# (V_1,V_2) &= \left( (W_1 \setminus B) \cup (V_1 \setminus C), (W_2 \setminus B) \cup (V_2 \setminus C) \right) \\
\end{align*}\]

We may think of this operation as taking boundary connected sums of the component handle bodies:

\[\begin{align*}
(W_1,W_2) \# (V_1,V_2) &= \left( W_1 \setminus_{\partial 1} V_1, W_2 \setminus_{\partial 2} V_2 \right) \\
\end{align*}\]

Note: We adopt the usual convention that \(\varphi\) is orientation reversing with respect to the induced orientations on \(\partial B\) and \(\partial C\).

The operation of connected sum of Heegaard decompositions has a particularly simple description in terms of the Heegaard models of section 1(b). If \(M^3\) is identified with \(|(W,\partial)|\) and \(N^3\) with \(|(V,\partial)|\) so that:

- \(h: (\partial W, \partial C, 0) \rightarrow (-\partial W, \partial C, 0)\) or. pres
- \(g: (\partial V, \partial P, 0) \rightarrow (-\partial V, \partial P, 0)\) or. pres

and if, by abuse of notation, we denote the corresponding Heegaard decompositions by \((W,h)\) and \((V,g)\), then:
\((W, h) \# (V, g) = (W \# V, h \# g)\)

where the boundary connect sum identifies \((D, 0)\) and \((E, p)\) and \(h \# g\) is well defined by assumption.

Note: We choose the new disc and base point as in Figure 10.

![Diagram](image)

**Figure 10**

(d) The genus one decomposition of \(S^3 \# (T, T_2)\)

Of particular interest to us is the genus one Heegaard decomposition of \(S^3 \# (T_1, T_2)\) where \(T_1\) is the standard torus embedded in \(R^3\) and \(S^3\) is \(R^3 \cup \{s\}\). (Actually there is only one genus one Heegaard decomposition of \(S^3\). [WJ.]

**Figure 11**

It is evident from Figure 7 that Singer's move of adding an unknotted one-handle is equivalent to forming the connect sum with the genus-one Heegaard decomposition of \(S^3\):

\(\phi(W_1, W_2) = (W_1 \# W_2) \# (T_1, T_2)\).

The genus one Heegaard decomposition of \(S^3\) has the following Heegaard model.
CHAPTER III: REPRESENTATION SPACES ASSOCIATED TO HEegaRD DECOMPOSITIONS

1. The diagram of representation spaces

(a) The diagram of representation spaces

Given an Heegard decomposition \((W_1, W_2)\) of a closed oriented 3-manifold \(M^3\), we have an associated commutative diagram of inclusions:

\[
\begin{array}{cccc}
(W_1,0) & \overset{i}{\hookrightarrow} & (F,0) & \overset{k}{\rightarrow} & (W_1,0) \\
\downarrow & & \downarrow & & \downarrow \\
(W_2,0) & \rightarrow & (M^3,0) & \rightarrow & (W_2,0)
\end{array}
\]

where:

\[
F = \partial W_1 = \partial W_2 = \partial W_1 \cap W_2
\]

\(F^\#\) is as in section II.1(a).

Note that each inclusion except for \(i\) is obtained by attaching 2 and 3 cells. Hence, by applying \(\Pi_1\), we obtain a commutative diagram of groups where all homomorphisms except \(i_*\) are surjective. "All the fundamental group is in the punctured surface":

\[
\Pi_1(F,0) \overset{i_*}{\rightarrow} \Pi_1(W_1,0) \rightarrow \Pi_1(W_2,0) \rightarrow \Pi_1(M^3,0)
\]

Clearly, \(i_*\) is an inclusion.

Finally, by applying the representation functor \(R_i\), we obtain a commutative diagram of spaces:

\[
\begin{array}{cccc}
\Pi_1(W_1,0) & \overset{i_*}{\rightarrow} & \Pi_1(F,0) & \rightarrow & \Pi_1(M^3,0)
\end{array}
\]

Figure 12

Choose a homeomorphism of \((T,E,p)\) which fixes \(E\) pointwise and interchanges the simple closed curves \(a\) and \(b\):

\[
\tau : (T,E,p) \rightarrow (-T,-E,p)
\]

\[
\tau_* : \Pi_1(T^e,p) \rightarrow \Pi_1(T^e,p)
\]

\[
\begin{array}{cccc}
a & \mapsto & b \\
b & \mapsto & a
\end{array}
\]

Clearly, this yields the desired Heegard model:

\[
S^3 = |(T,\tau)|
\]

\((T,\tau) = \text{the genus one Heeg. dec. of } S^3\).
Hence, we can consider their intersection product in $R^2$. We shall prove the following proposition.

**Proposition 1.1:**

(a) \[ \langle Q_1, Q_2 \rangle_{R^2} = 0 \text{ iff } \beta_1(M^3) \neq 0 \]

(b) \[ |\langle Q_1, Q_2 \rangle_{R^2}| = |H_1(M^3, \mathbb{Z})| \text{ iff } \beta_1(M^3) = 0 \]

(c) \[ Q_1 \cap Q_2 \text{ at } 1 \text{ if } \beta_1(M^3) = 0 \]

**Note:** We have denoted the top class of $R^2$ by $R^2$.

Propositions 1.3.1, 1.3.2 and 1.3.3 thru 9 are the key tools in the proof. We need to develop some background before beginning the proof.

Identify $M^3$ with an Heegaard model $(W, h)$. The diagram of inclusions in part (a) may be written as follows:

\[
\begin{array}{cccccccc}
(S^1, 0) & \overset{i}{\rightarrow} & (F^2, 0) & \overset{i}{\rightarrow} & (F, 0) & \overset{i}{\rightarrow} & (W, 0) & \rightarrow (M^3, 0) \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow h \\
(-S^1, 0) & \overset{i}{\rightarrow} & (-F^2, 0) & \overset{i}{\rightarrow} & (-F, 0) & \overset{i}{\rightarrow} & (-W, 0) & \rightarrow (M^3, 0)
\end{array}
\]

where:

\[ j : (W, 0) \longrightarrow [W, h] \]

\[ \times \longrightarrow [(W, 1)] \]

\[ k : (-W, 0) \longrightarrow [(-W, h)] \]

\[ \times \longrightarrow [(-W, 2)] \]

From this, we obtain the corresponding diagram of representation spaces:

\[ 48 \]

\[ 49 \]
\[ R(\Pi_1(F^*, 0)) \hookrightarrow R(\Pi_1(F, 0)) \hookrightarrow R(\Pi_1(M^*, 0)) \]
\[ R(\Pi_1(F^*, 0)) \hookrightarrow R(\Pi_1(F, 0)) \hookrightarrow R(\Pi_1(M^*, 0)) \]

\[ = R(h^*_g) \quad = R(h^*_g) \quad = R(h^*_g) \]

Considering all the injections as inclusions, we may make the identifications below:

1. \( R^* = R(\Pi_1(F^*, 0)) = (S^3)^* \)
\[ \rho \mapsto (\rho(a_i), \ldots, \rho(h_i), \ldots, \rho(b_i)) \]

2. \( R = R(\Pi_1(F)) = (\{A_1, \ldots, A_g, B_1, \ldots, B_g\} | [A_1, B_1] \cdots [A_g, B_g] = 1) \)

3. \( Q_1 = R(\Pi_1(W)) = (\{A_1, \ldots, A_g, 1, \ldots, 1\}) = (S^3)^* \)

4. \( Q_2 = R(\Pi_1(W_2)) = h^*_q(Q_1) = (S^3)^* \)

5. \( R(\Pi_1(M^*)) = Q_1 \cup Q_2 \quad \text{(under identifications)} \)

In particular, we have fixed orientations on \( R^* \) and \( Q_1 \).

**Proof of Proposition 1.1:**

Consider the natural cocycles in \( H^2(R^*, Z) \):
\[ \alpha_1 = z^*(c_1) \quad \beta_1 = z^*(d_1) \quad i = 1, \ldots, g \]

If we consider \( R^* \) as having the induced orientation via the ordered basis \( (c_1, \ldots, c_g, d_1, \ldots, d_g) \), the dual \( 3n - 3 \) cycles are given by:
\[ \det(\alpha_1) = (-1)^{\frac{n}{2}}(\rho | A_1 = 1) \quad \det(\beta_1) = (-1)^{\frac{n}{2}}(\rho | B_1 = 1) \]

Notes:

1. \( \rho \mid A_1 = 1 \) is oriented by \( z \times \cdots \times z \times 1 \times z \times \cdots \times z \times z \times \cdots \times z \)

2. Similarly, \( \rho \mid B_1 = 1 \) has the obvious orientation.

Clearly:
\[ Q_1 = D(\beta_1 \cup \cdots \cup \beta_g) \]

Then:
\[ \langle Q_1, Q_2 \rangle_{R^*} = \langle Q_1, (h^*_g)(Q_1) \rangle_{R^*} = \langle \beta_1 \cup \cdots \cup \beta_g \cup (h^*_g)^*(\beta_1) \cup \cdots \cup (h^*_g)^*(\beta_g), w \rangle \]

where \( w \) is the top class of \( R^* \). By Proposition 1.3.1, the matrix for \( (h^*_g)^* \) with respect to the ordered basis \( (a_1, \ldots, a_g, b_1, \ldots, b_g) \) is the same as the matrix for \( h^*_g \) with respect to \( (a_1, \ldots, a_g, b_1, \ldots, b_g) \):
\[ (h^*_g)^* = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = h^*_g, \quad B = (b_{ij}) \]

Using the fact that \( \beta_1 \cup \beta_1 = 0 \):
\[ \langle Q_1, Q_2 \rangle_{R^*} = \langle \beta_1 \cup \cdots \cup \beta_g \cup \sum_{\sigma_1} b_{\sigma_1} a_{\sigma_1} \cup \cdots \cup \sum_{\sigma_g} b_{\sigma_g} a_{\sigma_g} \cup \beta_1 \cup \cdots \cup \beta_g, w \rangle = \sum_{\sigma_1} b_{\sigma_1} \cdots b_{\sigma_g} \langle \beta_1 \cup \cdots \cup \beta_g \cup a_{\sigma_1} \cup \cdots \cup a_{\sigma_g}, w \rangle \]
\[ \langle Q_1, Q_2 \rangle_{R^*} = (-1)^{\frac{n}{2}} \sum_{\sigma_1} b_{\sigma_1} \cdots b_{\sigma_g} \langle a_{\sigma_1} \cup \cdots \cup a_{\sigma_g}, \beta_1 \cup \cdots \cup \beta_g, w \rangle = (-1)^{\frac{n}{2}} \sum_{\sigma_1} b_{\sigma_1} \cdots b_{\sigma_g} (-1)^{\frac{n}{2}} \langle a_{\sigma_1} \cup \cdots \cup a_{\sigma_g}, \beta_1 \cup \cdots \cup \beta_g, w \rangle \]
But, as established in section III.3(d), \( \beta_1 \cdots \beta_g \) is dual to \( w \):

\[
\langle Q, Q \rangle_{R^g} = (-1)^{k_1} \text{det}(B).
\]

Hence, by comparison with section II.1(c):

\[
\beta_1(M) \neq 0 \iff \langle Q_1, Q_2 \rangle_{R^g} = 0.
\]

For the same reason:

\[
|\langle Q_1, Q_2 \rangle_{R^g}| = |\beta_1(M, X)| \iff \beta_1(M, X) = 0.
\]

We turn now to the proof of part (c). Using the same models as above, we may make the identifications:

\[
\begin{align*}
(6) & \quad H_1(\mathbb{R}^g, X) = \langle c_1, \ldots, c_g, d_1, \ldots, d_g \rangle \cong \mathbb{Z}^{2g} \\
(7) & \quad T_1(\mathbb{R}^g) = H_1(\mathbb{R}^g, X) \otimes \mathbb{S} \\
(8) & \quad T_1(Q_1) = \langle c_1, \ldots, c_g \rangle \otimes \mathbb{S} \\
(9) & \quad T_1(Q_2) = D_2(R_{h_2})(T_1(Q_1))
\end{align*}
\]

Transversality of \( Q_1 \) and \( Q_2 \) at 1 is equivalent to the equation:

\[
T_1(Q_1) + T_1(Q_2) = T_1(\mathbb{R}^g).
\]

(10) But \( T_1(Q_1) + T_1(Q_2) \) is the image of the map:

\[
T_1(Q_1) \otimes T_1(Q_2) \xrightarrow{i \otimes D_1(R_{h_2})} T_1(\mathbb{R}^g).
\]

(11) The direct product here is "abstract" and we identify the second factor as:

\[
T_1(Q_1) = \langle s_1', \ldots, s_g' \rangle \otimes \mathbb{S}.
\]

(12) By Proposition 1.3.3, the map is given by the matrix:

\[
i \otimes D_1(R_{h_2}) = \begin{bmatrix} 1 & A^t \\ 0 & B \end{bmatrix} \otimes \text{id}.
\]

(13) By the above remarks, \( Q_1 \neq Q_2 \) at 1 iff \( i \otimes D_1(R_{h_2}) \) is nonsingular. Clearly, this is equivalent to the nonsingularity of \( B \). The result follows as before. \( \square \)

Note: Apparently, the fact that \( h_2 \) is symplectic is important here.

2. The boundary map \( \partial \) and induced orientations

(a) The boundary map \( \partial \)

Consider:

\[
(\mathbb{R}^g, 0) \longrightarrow (\mathbb{R}^g, 0).
\]

This induces a map of representation spaces, the restriction map:

\[
R(\mathbb{R}_1(\mathbb{R}^g, 0)) \longrightarrow R(\mathbb{R}_1(\mathbb{R}^g, 0)).
\]

We denote this map as the boundary map \( \partial \):

\[
\partial : \mathbb{R}^g \longrightarrow R(\mathbb{R}_1(\mathbb{R}^g, 0)).
\]

We have the following properties of \( \partial \) ([Sh], [I]):

**Proposition 2.1:** (1) \( \partial \) is surjective

(2) \( \text{crit} \text{pts of } \partial = \text{reducible reps} \)
proof of (1):

As in section 1(a):

\[ R^8 = (S^3)^2 \mathbb{R} = \{ (A_1, B_1, \ldots, A_g, B_g) \mid A_i, B_i \in S^3 \} \]

and \( R(\Pi_1(F^8,0)) \) is identified with \( S^3 \).

This latter identification may be chosen so that the map \( j \) may be written as:

\[ j : (S^3)^2 \mathbb{R} \rightarrow S^3 \]
\[ \begin{array}{c}
(A_1, B_1, \ldots, A_g, B_g) \\
\rightarrow [A_1, B_1] \cdots [A_g, B_g]
\end{array} \]

(1)

(2) Let:

\[ B_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

(3) Then:

\[ j(\Xi, \nu, \lambda, I, \ldots, I) = R_{\theta} \]

(4) If \( A \in S^3 \), then \( A = CR_{\theta}C^{-1} \) for some \( \theta, C \in S^3 \).

(5) Hence:

\[ A = j(CR_{\theta}C^{-1}, C\sigma C^{-1}, I, \ldots, I) \]

Proof of (2):

Let:

\[ \rho = (A_1, B_1, \ldots, A_g, B_g) \in R^8 \]

\[ \bar{\rho} = [A_1, B_1] \cdots [A_g, B_g] \in R(\Pi_1(F^8,0)) \]

(6) Of course:

\[ T_\rho(R^8) = T_{A_1}(S^3) \otimes T_{B_1}(S^3) \otimes \cdots \otimes T_{A_g}(S^3) \otimes T_{B_g}(S^3) \]
\[ T_{\bar{\rho}}(R(\Pi_1(F^8,0))) = T_{\bar{\rho}}(S^3) \]

(7) By computing derivatives as before and applying appropriate left and right translations to each factor of the given splitting of \( T_\rho(R^8) \) and to \( T_{\bar{\rho}}(S^3) \), we can construct a commutative diagram:

\[ T_\rho(R^8) = T_{A_1}(S^3) \otimes \cdots \otimes T_{B_g}(S^3) \xrightarrow{\Gamma_{\rho}(\bar{\rho})} T_{\bar{\rho}}(S^3) \]
\[ T_{\bar{\rho}}(S^3) \xrightarrow{F} T_{\bar{\rho}}(S^3) \]

where:

\[ F(X_1, Y_1, \ldots, X_g, Y_g) = \sum_{j=1}^g \left( X_j - \sum_{j=1}^g X_j A_j B_j^{-1} A_j^{-1} B_j^{-1} \right) \]
\[ + \sum_{j=1}^g \left( Y_j - \sum_{j=1}^g Y_j A_j B_j^{-1} A_j^{-1} B_j^{-1} \right) \]

(8) Since the following elements of \( \Pi_1(F^8,0) \) are a free basis:

\[ \left( i_j A_j b_j a_j^{-1} b_j a_j^{-1}^{-1} i_j - 1 A_j b_j a_j^{-1} b_j a_j^{-1}^{-1} i_j \right) \]

we see that Proposition 2.1(2) is equivalent to the following claim.

Claim: Let \( (C_1, b_1, \ldots, C_g, b_g) \in (S^3)^2 \mathbb{R} = R^8 \). Let:

\[ F : T_{\bar{\rho}}(S^3) \otimes \cdots \otimes T_{\bar{\rho}}(S^3) \rightarrow T_{\bar{\rho}}(S^3) \]

where:

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\[ 55 \]
\( F(X_1, Y_1, \ldots, X_n', Y_n') = \sum_{j=1}^g (X_j - D_j X_j D_j^{-1}) + \sum_{j=1}^g (Y_j - C_j Y_j C_j^{-1}) \).

Then \( F \) is surjective if and only if \((C_1, \ldots, D_n)\) is irreducible.

(9) Hence, we begin by computing the image of the map:

\[
L : T_1(S^2) \longrightarrow T_1(S^2) \]
\[
X \longmapsto X - BXB^{-1}.
\]

(10) Suppose:

\[ D = R \theta, \quad X = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \]

Then:

\[
L(X) = \begin{bmatrix} 0 & e^{i \theta} \\ e^{-i \theta} & 0 \end{bmatrix}
\]

Hence:

Image(L) = \{ 0 \} \quad \text{if} \quad D = \pm I

= \mathbb{S}_0 = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \quad \text{if} \quad D \neq \pm I.

(11) In general, \( D = BR \theta B^{-1} \) and:

Image(L) = \{ 0 \} \quad \text{if} \quad D = \pm I

= \mathbb{S}_0 \quad \text{if} \quad D \neq \pm I.

(12) Hence, let \( C_j = E_{2j-1} R_{\theta_j} E_{2j-1}^{-1}, D_j = E_{2j} R_{\theta_j} E_{2j}^{-1} \), so:

Image(F) = \sum (E_j \mathbb{S}_0 E_j^{-1} | E_j R_{\theta_j} E_j^{-1} \neq \pm I).

(13) Since \( \dim(S^1) = 2 \), we conclude that:

\( F \) is surjective if and only if there exists \( j, k \) with

\[ 1 \neq j \neq k \leq 2g \]

such that:

(a) \( E_j R_{\theta_j} E_j^{-1} \neq \pm I \)

(b) \( E_k R_{\theta_k} E_k^{-1} \neq \pm I \)

and (c) \( E_j R_{\theta_j} E_j^{-1} \neq \pm I \).

(14) Let \( F_j = E_j R_{\theta_j} E_j^{-1}, F_k = E_k R_{\theta_k} E_k^{-1}, F_j \neq \pm I \neq F_k \).

Suppose \( \langle F_j, F_k \rangle \) is reducible. Then there exists \( K \in S^3 \) such that:

\[ E^{-1} F_j E^{-1} = F_{\theta_j} \]
\[ E^{-1} F_k E^{-1} = F_{\theta_k} \]

So:

\[ E^{-1} R_{\theta_j} E^{-1} = R_{\theta_j} \]
\[ E^{-1} R_{\theta_k} E^{-1} = R_{\theta_k} \]

So:

\[ \mathbb{S}_0 \in \text{Normalizer}(\langle F_j \rangle) = \text{Normalizer}(S^1) \]
\[ \mathbb{S}_0 \in \text{Normalizer}(\langle F_k \rangle) = \text{Normalizer}(S^1) \]

So:

\[ E_j^{-1} F_k \in \text{Normalizer}(S^1) \]

Claim: \( \langle F_j, F_k \rangle \) is reducible if and only if \( E_j^{-1} R_{\theta_j} E_{\theta_j} \in \text{Norm}(S^1) \), assuming \( F_j \neq \pm I \neq F_k \).

Subproof: We have just shown the necessity. Now suppose \( B = E_j^{-1} R_{\theta_j} E_{\theta_j} \in S^1 \).

Then:
\[ E_j E_k^{-1} = H_{jk} \]
\[ E_j E_k E_j^{-1} = E_j^{-1} E_k H_{jk} E_j^{-1} E_j = H_{jk} \]

In other words, \( \langle F_j, F_k \rangle \) is diagonalizable. \( \square \)

(15) Finally, it is a straightforward exercise to verify that:
\[ E S_n E^{-1} = S_n \text{ if } E \in \text{Normalizer}(S^n) \]

The result follows immediately from (13) and (15). \( \square \)

(b) The singular set \( S \)

Let:
\[ S = \{ \rho : \rho : \Pi_1(F^*, 0) \rightarrow S^n \text{ reducible} \} \]

By Proposition 1.2.1, every reducible representation is conjugate to a diagonal representation. Hence, every reducible representation satisfies the generating relation for \( \Pi_1(F, 0) \) and we conclude that:
\[ S \subset \hat{R} \quad S = S(\Pi_1(F, 0)) = S(\Pi_1(F^*, 0)) \]

As an immediate corollary to Proposition 2.1 (and the definition of \( R \) and \( \hat{R} \)) we conclude:

Corollary 2.2: (a) \( R = S^{-1}(1) \)

(b) \( R \backslash S \) is an open smooth manifold of dimension \( 6g - 3 \).

Appealing to Proposition 1.2.1 and the naturality of the \( SO_3 \) action, we deduce the existence of the following free actions:

\[ SO_3 \times (R \backslash S) \longrightarrow (R \backslash S) \text{ freely} \]
\[ SO_3 \times (R \backslash S) \longrightarrow (R \backslash S) \text{ freely} \]
\[ SO_3 \times (Q \backslash S) \longrightarrow (Q \backslash S) \text{ freely} \]
\[ SO_3 \times (Q \backslash S) \longrightarrow (Q \backslash S) \text{ freely} \]

Let:
\[ \hat{R} = (R \backslash S)/SO_3 \]
\[ \hat{Q}_i = (Q \backslash S)/SO_3 \]
\[ \hat{Q}_i = (Q \backslash S)/SO_3 \]

From the above observation, we derive:

Proposition 2.3:

(a) \( \hat{R} \) is a smooth open manifold of dimension \( 6g - 3 \)

(b) \( \hat{Q}_i \) is a smooth open manifold of dimension \( 3g - 3 \) \( i = 1, 2 \).

(c) Induced orientations

Consider the following choices of orientation:
\[ \hat{W}_i \text{ - induced orientation from } \hat{M} \quad i = 1, 2 \]
\[ F \text{ - induced orientation from } \hat{W}_i \]
\[ F^* \text{ - induced orientation from } F \]
\[ \hat{F}^* \text{ - induced orientation from } \hat{F}^* \]

The orientation on \( \hat{F}^* \) determines a unique generator of \( \Pi_1(F^*, 0) \). We adopt the following convention:
$R_3$ — induced orientation from the basis of $\mathbb{R}^3(0,0)$ corresponding to the orientation on $\mathbb{R}^3$.

Recall that a short exact sequence of oriented vector spaces:

$$0 \longrightarrow U \overset{\alpha}{\longrightarrow} V \overset{\beta}{\longrightarrow} W \longrightarrow 0$$

$[(u_1, \ldots, u_p)] = \text{orientation on } U$

$[(v_1, \ldots, v_q, v_{q+1}, \ldots, v_{q+p})] = \text{orientation of } V$

$[(w_1, \ldots, w_q)] = \text{orientation on } W$

is said to be compatibly oriented if we can assume:

$$v_{q+j} = \alpha(u_j), \quad i \neq j \neq p$$

$$w_j = \beta(v_j), \quad 1 \leq j \leq q$$

Given a short exact sequence of vector spaces with two of $U, V, W$ oriented, there is a unique orientation on the third making the sequence compatibly oriented. We shall refer to this orientation as the induced orientation.

Next, we orient $R \setminus S$ by the following convention. Consider the sequence:

$$R \setminus S \longrightarrow R^3 \longrightarrow R_3$$

By Corollary 2.2 and Proposition 2.1, we conclude that if $p \in R \setminus S$, then:

$$0 \longrightarrow T_p(R \setminus S) \longrightarrow T_p(R^3) \longrightarrow T_1(R_3) \longrightarrow 0$$

is a short exact sequence.

Now, given an arbitrary orientation of $R^3$, we orient these vector spaces as follows:

$$T_1(R_3) \quad \text{— induced orientation from orientation on } R_3$$

$$T_p(R^3) \quad \text{— induced orientation from } R^3$$

$$T_p(R \setminus S) \quad \text{— induced orientation from } R \setminus S$$

It is easy to see that this gives a well-defined continuous family of orientations on the tangent spaces of $R \setminus S$ and, hence, a well-defined orientation on $R \setminus S$.

In a similar manner, we may orient $R$. More precisely, we consider the sequence:

$$SO_3 \longrightarrow R^3 \longrightarrow R \quad p \in R \setminus S$$

and the associated short exact sequence:

$$0 \longrightarrow T_p(SO_3) \longrightarrow T_p(R \setminus S) \longrightarrow T_p(R) \longrightarrow 0$$

Since $SO_3$ acts freely on $R^3 \setminus S$, we have a natural homeomorphism:

$$SO_3 \longrightarrow SO_3 \cdot p$$

$$A \longmapsto A \cdot p$$

which induces an orientation on $SO_3 \cdot p$. (Recall that $S^3$ has a fixed orientation. $SO_3$ is oriented as the quotient $S^3/S^2$.)

We orient the vector spaces above as follows:

$$T_p(SO_3 \cdot p) \quad \text{— induced orientation from } SO_3 \cdot p$$

$$T_p(R \setminus S) \quad \text{— induced orientation from } R \setminus S$$

$$T_p(R) \quad \text{— induced orientation from } T_p(SO_3 \cdot p) \text{ and } T_p(R \setminus S)$$
again, it is easy to see that we obtain in this manner a well-defined orientation on \( \hat{\mathcal{E}} \).

Finally, in the same manner, given an arbitrary orientation on \( Q_1 \), we obtain an orientation on \( \hat{Q}_1 \):

\[
\begin{align*}
T_{\rho}(SO_2) & \quad - \text{induced orientation from } SO_2, \\
T_{\rho}(Q_4) & \quad - \text{induced orientation from } Q_4, \\
T_{\rho}(Q_4) & \quad - \text{induced orientation from } T_{\rho}(SO_2), T_{\rho}(Q_4). \\
\end{align*}
\]

Note: Under our conventions, the only fixed orientations are those of \( E_2 \) and the orbits of \( SO_2 \), \( R \setminus S \) and \( R \setminus S \) and \( \hat{\mathcal{E}}, Q_4 \setminus S \) and \( \hat{Q}_4 \) need only be compatibly oriented with respect to these. The motivation for these conventions will be seen shortly.

CHAPTER IV: CASSON’S IN Variant FOR ORIEN TED HOMOLOGY 3–SPHERES

1. Casson’s invariant for Heegaard decompositions

(a) The intersection products

Let \( M^3 \) be an homology 3–sphere:

\[
\begin{align*}
H_0(M^3,\mathbb{Z}) &= H_3(M^3,\mathbb{Z}) = \mathbb{Z}, \\
H_1(M^3,\mathbb{Z}) &= H_2(M^3,\mathbb{Z}) = \{0\}. \\
\end{align*}
\]

Let \((W_1, W_2)\) be a Heegaard decomposition of \( M^3 \). As an immediate consequence of Proposition III.1.1, we obtain:

Proposition 1.1:

(a) \( \langle Q_4, Q_4 \rangle \mid G_2 = 1 \)

(b) \( Q_4 \cap Q_2 \) at 1

By Proposition 1.2.1, every reducible representation is conjugate to a diagonal representation. Hence, every reducible representation of \( H_1(M^3,\mathbb{Z}) \) factors through \( H_1(M^3,\mathbb{Z}) \). We deduce the following corollary:

Corollary 1.2:

(a) \( S(H_1(M^3,\mathbb{Z})) = R(H_1(M^3,\mathbb{Z})) \cap S = Q_4 \cap Q_2 \cap S = \{1\} \).

(b) \( Q_4 \setminus S \cap (Q_2 \setminus S) \) is compact.

(c) \( \hat{Q}_4 \) and \( \hat{Q}_2 \) are properly embedded open submanifolds of \( \hat{\mathcal{E}} \).

(c) \( \hat{Q}_4 \cap \hat{Q}_2 \) is compact.

Note: If the genus of the Heegaard decomposition is one, then \( H_1(F,\mathbb{Z}) \) is abelian. It follows from Proposition 1.2.3 that:

\[
\begin{align*}
R = R(H_1(F,\mathbb{Z})) = S & \quad \text{if genus} = 1, \\
\hat{R} = \hat{Q}_4 = \hat{Q}_2 = \varnothing & \quad \text{if genus} = 1.
\end{align*}
\]

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Choose an isotopy of $\hat{R}$ with compact support which carries $\hat{\mathcal{Q}}_2$ to $\hat{\mathcal{Q}}_2$ where $\hat{\mathcal{Q}}_2$ is transverse to $\hat{\mathcal{Q}}_1$. Then, by Proposition III.2.3:

$$(\hat{\mathcal{Q}}_2) \text{ is isotopic to } \hat{\mathcal{Q}}_2$$
$$\hat{\mathcal{Q}}_1 \cap \hat{\mathcal{Q}}_2 = \text{ finite set of points}.$$ 

Given orientations on $\hat{\mathcal{Q}}_1$ and $\hat{\mathcal{Q}}_2$, we can define the algebraic intersection $\langle \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2 \rangle_R$:

$$\langle \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2 \rangle_R = \Sigma \langle \text{sign}(\rho; \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2) | \rho \in \hat{\mathcal{Q}}_1 \cap \hat{\mathcal{Q}}_2 \rangle$$
$$[(u_1, \ldots, u_{2g-3})] = \text{ ind. or. of } T_\rho \hat{\mathcal{Q}}_1$$
$$[(v_1, \ldots, v_{2g-3})] = \text{ ind. or. of } T_\rho \hat{\mathcal{Q}}_2$$
$$[(w_1, \ldots, w_{2g-3})] = \text{ ind. or. of } T_\rho \hat{\hat{\mathcal{Q}}_1}$$
$$[(w_1, \ldots, w_{2g-3})] = \text{ sign}(\rho; \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2) [(u_1, \ldots, u_{2g-3}, v_1, \ldots, v_{2g-3})]$$

By standard arguments, this is well-defined.

(b) Casson's invariant for Heegaard decompositions

If $M^3$ is an oriented homology 3-sphere with an Heegaard decomposition $(W_1, W_2)$, we define:

$$\lambda(M^3; W_1, W_2) = \frac{\langle \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2 \rangle_R}{2(\mathcal{Q}_1, \mathcal{Q}_2)_{\hat{R}}^R} \cdot S$$

where all the orientations are chosen by the conventions of section III.2(c), and $S$ is either $-1$ or $1$. $S$ will be determined in section V.5.

Notes: 1) $\lambda(M^3; W_1, W_2)$ is well-defined by the compatibility conventions of section III.2(c). For instance, if we change the orientation of $R^3$, we must change that of $R \setminus S$ and, hence, that of $\hat{R}$. Hence, by Proposition 1.1, we can always choose the orientations subject to the convention:

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle^R_{\hat{R}} = 1.$$ 

2) If $g = 1$, we let $\lambda(M^3; W_1, W_2) = 0$.

If we change the orientation of $M^3$, this changes the orientation of $\partial M^3$. This changes the generator $a$ of $\pi_1(\partial M^3)$ to $a^{-1}$. The induced orientations on $R_3$ come, respectively, from the identifications:

$$R_3 \xrightarrow{\pi} S^3 \quad R_3 \xrightarrow{\pi} S^3$$

Since inversion changes the orientation of $S^3$, we see that the induced orientation of $R_3$ must change. We may maintain the given orientation on $R^3$ by 1). In order to maintain compatibility, the orientation on $R \setminus S$ and, hence, on $\hat{R}$ must change. ($SO_3$ has a fixed orientation). Therefore:

$$\lambda(-M^3; W_1, W_2) = -\lambda(M^3; W_1, W_2).$$
c) Independence of Heegard decomposition

We wish to show that \( \lambda(M^3; W_1, W_2) \) is independent of the Heegard decomposition \( (W_1, W_2) \). It is evident that this invariant is an invariant of ambient isotopy of Heegard decompositions. Hence, from Singer’s Theorem, we conclude that it suffices to show that it is also invariant under stabilization, i.e., under addition of an unknotted one-handle.

If \((W, h)\) is a Heegard model, let:

\[
\lambda(W, h) = \lambda(I(W, h)); \quad W = \{1\}, \ W = \{2\}.
\]

Proposition 1.3: Given a Heegard model \((W, h)\), let:

\[
W' = W \# T \quad h' = h \# \tau.
\]

Then:

\[
\lambda(W', h') = \lambda(W, h).
\]

Proof: Choose a basis \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) for \( \Pi_1(F, \partial) \) as in section II.1(a). Let \( a_{g+1} = a, \ b_{g+1} = b \) as in section II.2(d):

\[
\begin{align*}
\Pi_1(F, \emptyset) &= \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \\
\Pi_1(F', \emptyset) &= \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \\
\Pi_1(F, \partial) &= \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle \sum_{j=1}^{g+1} \ [a_j, b_j] = 1 \\
\Pi_1(F', \partial) &= \langle a_1, \ldots, a_g, b_1, \ldots, b_g, a_{g+1}, b_{g+1} \rangle \sum_{j=1}^{g+1} \ [a_j, b_j] = 1 \\
\Pi_1(Q, \emptyset) &= \langle a_1, \ldots, a_g \rangle \\
\Pi_1(Q', \emptyset) &= \langle a_1, \ldots, a_g, a_{g+1} \rangle.
\end{align*}
\]

We may make the following identifications:

\[
\begin{align*}
(R')^* &= (S^3)^{g+1} = \{(A_1, \ldots, A_g, B_1, \ldots, B_g, A_{g+1}, B_{g+1})\} \\
R^* &= (S^3)^{g+1} = \{(A_1, \ldots, A_g, B_1, \ldots, B_g)\} \\
R' &= \{(A_1, \ldots, A_g, B_1, \ldots, B_g, A_{g+1}, B_{g+1})\} \bigwedge_{j=1}^{g+1} [A_j, B_j] = 1 \\
R &= \{(A_1, \ldots, A_g, B_1, \ldots, B_g)\} \bigwedge_{j=1}^{g} [A_j, B_j] = 1 \\
Q'_1 &= \{(A_1, \ldots, A_g, I, \ldots, I, A_{g+1}, I)\} \\
Q_1 &= \{(A_1, \ldots, A_g, I, \ldots, I)\} \\
Q'_2 &= \{(h')^* (Q'_1)\} \\
Q_2 &= \{(h')^* (Q_1)\}.
\end{align*}
\]

Then:

\[
\lambda(W', h') = \frac{(-1)^{g+1} Q'_1 \partial_1 \partial_2 R}{2 \langle Q_1, Q_2 \rangle S}.
\]

We wish to compare the various terms.

(i) Comparison of denominators

From the above identifications, we observe that:

\[
\begin{align*}
(R')^* &= R^* \times S^3 \times S^3 \\
Q'_1 &= Q_1 \times S^3 \times \{I\} \\
Q'_2 &= Q_2 \times \{I\} \times S^3
\end{align*}
\]
This last identity follows from the definition of \( \tau \) given in section II.2(d).

\[ H_k(S^3) \otimes \cdots \otimes H_k(S^3) \otimes \cdots \otimes H_k(S^3) \xrightarrow{\cup} H_k(R^k) \]

\[ H_k(R^k) \otimes \cdots \otimes H_k(R^k) \otimes H_k(R^k) \xrightarrow{\cup} H_k((R')^k) \]

\[ H_k(R^k) \otimes \cdots \otimes H_k(R^k) \otimes H_k(R^k) \otimes \cdots \otimes H_k(R^k) \xrightarrow{\cup} H_k(R^k) \]

\[ H_k^*(S^3) \otimes H_k^*(S^3) \otimes H_k^*(S^3) \xrightarrow{\cup} H_k^*((R')^k) \]

\[ H_k^*(R^k) \otimes H_k^*(R^k) \otimes H_k^*(R^k) \xrightarrow{\cup} H_k^*((R')^k) \]

\[ H_k^*(R^k) \otimes \cdots \otimes H_k^*(R^k) \otimes H_k^*(R^k) \xrightarrow{\cup} H_k^*((R')^k) \]

In the proof of Proposition III.1.1, we consider the natural 3-cocycles in \( * \) and \( (R')^k \):

\[ \alpha_1 = \varepsilon^*(c_1) \quad \beta_1 = \varepsilon^*(d_1) \quad i = 1, \ldots, g \]

\[ \alpha_1 = \varepsilon^*(c_1) \quad \beta_1 = \varepsilon^*(d_1) \quad i = 1, \ldots, g^* + 1 \]

Then:

\[ \alpha_1 = \alpha_1 \times 1 \quad \beta_1 = \beta_1 \times 1 \quad 1 \neq i \neq g \]

\[ \alpha_{g+1} = \alpha \times 1 \quad \beta_{g+1} = \beta \times \beta \]

Let \( z \) and \( \mu \) be the top-class and coclass of \( S^3 \) as in section I.3(b).

Then:

\[ \beta_1 = (1 \times \cdots \times 1) \times \mu \times \cdots \times \mu_{2-1} \times \mu \times 1 \times \cdots \times 1 \quad \mu_j = 1 \]

\[ \alpha = \mu \times 1 \quad \beta = 1 \times \mu \]

Our conventions on orientations allow us to choose the following orientations:

\[ R^k : \quad w = (z \times \cdots \times z) \times (z \times \cdots \times z) \]

\[ (R')^k : \quad w' = w \times (z \times z) \]

\[ Q_1 : \quad w_1 = (z \times \cdots \times z) \times (1 \times \cdots \times 1) \]

\[ Q_1' : \quad w_1' = w_1 \times (z \times 1) \]

\[ Q_2 : \quad w_2 = (h^k)_g(w_1) \]

\[ Q_2' : \quad w_2' = w_2 \times (1 \times z) \]

Let:

\[ P_1 = (1 \times \cdots \times 1) \times (z \times \cdots \times z) \quad P_2 = (h^k)_g(P_1) \]

\[ P_1' = P_1 \times (1 \times z) \quad P_2' = P_2 \times (z \times 1) \]

From the above observations, we compute that:

\[ \langle Q_1, P_1 \rangle_{R^k} = 1 = \langle \beta_1, \cdots, \beta_g, P_1 \rangle \]

\[ \langle Q_1', P_1' \rangle_{(R')^k} = (-1)^{g+1} \langle \beta_1', \cdots, \beta_g', \beta_1 \times \cdots \times \beta_g, P_1 \rangle \]

\[ \langle Q_2, P_2 \rangle_{R^k} = 1 = \langle h^k(\beta_1, \cdots, \beta_g), P_2 \rangle \]

\[ \langle Q_2', P_2' \rangle_{(R')^k} = (-1)^{g+1} \langle h^k \beta_1, \cdots, \beta_g \rangle \langle \beta_1', \cdots, \beta_g', \beta_1 \times \cdots \times \beta_g, P_1 \rangle \]

As in the proof of Proposition III.1.1, we conclude:

\[ w_1 = D(\beta_1, \cdots, \beta_g) \]

\[ w_1' = (-1)^{g+1} D(\beta_1', \cdots, \beta_g', \beta_1 \times \cdots \times \beta_g) \]

\[ w_2 = D((h^k)_g(\beta_1, \cdots, \beta_g)) \]

\[ w_2' = (-1)^{g+1} D((h^k)_g(\beta_1, \cdots, \beta_g)) \times (1 \times z) \times (z \times 1) \]

\[ w_1' = w_1' \times (1 \times z) \times (z \times 1) \]
Let: \[ \delta = \beta_1 \cup \cdots \cup \beta_g. \]

Then:
\[
\langle Q'_1, Q'_2 \rangle_{(R')}^* = (-1)^{2g+1} \langle \delta \times \beta \rangle \cup (h^2(\delta) \times \alpha_x^w) , w' \rangle
\]
\[
= (-1)^{3g+1} \langle \delta \cup h^2(\delta) \rangle \times (\beta \cup w) , w \times z \times z \rangle.
\]

Hence:
\[
\langle Q'_1, Q'_2 \rangle_{(R')}^* = (-1)^{3g+2} \langle \delta \cup h^2(\delta) , w \rangle \cdot \langle \alpha \cup \beta , z \times z \rangle
\]
\[
= (-1)^{g} \langle \delta \cup h^2(\delta) , w \rangle.
\]

2.) \[ \langle Q'_1, Q'_2 \rangle_{(R')}^* = (-1)^{K} \langle Q_1, Q_2 \rangle_{R^*} \]

(ii) Comparison of numerators

We wish to carry out a similar comparison of numerators. Although \( \hat{\mathbf{R}} \) is not a factor of \( \hat{\mathbf{R}}' \), we shall see that \( \hat{\mathbf{R}} \) is a submanifold of \( \hat{\mathbf{R}}' \) and, hence, a "local factor". This will be sufficient, since the following considerations imply that all the intersections occurring in the numerators will take place in a regular neighborhood of \( \hat{\mathbf{R}} \).

By an argument similar to section III.2(b):
\[
R \times (I) \times (I) \subset R \times S^3 \times (I) \subset R'
\]
\[
R \times (I) \times (I) \subset R \times [I] \times S^3 \subset R'
\]
\[
S \times (I) \times (I) = S' \cap (R \times (I) \times (I)) \subset S'.
\]

By an obvious abuse of notation:
\[
R \setminus S = R' \setminus S'
\]
\[
Q_1 \setminus S = Q'_1 \setminus S'
\]
\[
Q_1 \cap Q_2 = Q'_1 \cap Q'_2 \subset R'.
\]

Since \( S \) acts by conjugation on each component, we obtain a well-defined commutative diagram:

\[
\begin{array}{ccc}
R \setminus S & \rightarrow & R' \setminus S' \\
\downarrow & & \downarrow \\
\hat{\mathbf{R}} & \rightarrow & \hat{\mathbf{R}}'
\end{array}
\]

from which we conclude that:
\[
\hat{\mathbf{R}} = \hat{\mathbf{R}}' \quad \hat{Q}_1 = \hat{Q}'_1
\]
\[
\hat{Q}_1 \cap \hat{Q}_2 = \hat{Q}'_1 \cap \hat{Q}'_2 = \hat{\mathbf{R}}.
\]

In order to compute the numerators, we need to isotope \( \hat{Q}_1 \) and \( \hat{Q}_2 \) to be transverse to \( \hat{\mathbf{R}} \), \( \hat{Q}_1 \), and \( \hat{Q}'_1 \), and then count intersections. We would like to construct a single isotomy of \( \hat{\mathbf{R}}' \) to accomplish this. Furthermore, we want this isotomy to be as simple as possible so that we can compare intersections and orientations. To see that this is possible, we investigate the regular neighborhood of \( \hat{\mathbf{R}} \) in \( \hat{\mathbf{R}}' \).

**R \setminus S is a properly embedded submanifold of R' \setminus S'**

In order to see this, consider the map:
\[
f : (R')^* \rightarrow S^3 \times S^3 \times S^3
\]
\[
\rho' \rightarrow (\hat{\mathbf{R}}', \hat{\mathbf{R}}, \mathbf{A}, \mathbf{B}).
\]

where \( \rho' = (\rho, \mathbf{A}, \mathbf{B}) \).

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Then:

\[ R = f^{-1}(I, I, I) \]

\[ R' = f^{-1}(\{I\} \times S^2 \times S^2). \]

Clearly:

\[
D_p(f) = \begin{pmatrix}
D_p(f) \cdot [A, B] & \ast & \ast \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\]

It follows from Proposition III.2.1 that \( R \setminus S \subset R' \setminus S' \) is a differentiable embedding. Moreover, from the observations above, this embedding is proper. Hence, we conclude that:

\[ R \setminus S \subset R' \setminus S' \text{ is a proper embedding} \]

\[ G \setminus S \subset G' \setminus S' \text{ is a proper embedding.} \]

In addition, we have the following commutative diagram of proper embeddings:

\[
\begin{array}{ccc}
R \setminus S & \rightarrow & (R \setminus S) \times S^2 \times \{I\} \\
\downarrow & & \downarrow \\
R' \setminus S' & \rightarrow & (R' \setminus S') \times S^2 \times \{I\}
\end{array}
\]

(The latter inclusions follow from the definition of \( R' \) and \( S' \).)

\( \tilde{R} \) is a properly embedded submanifold of \( \tilde{R}' \)

Since all the proper embeddings above are invariant under the \( S_0^2 \) action, we conclude immediately that:

\[ \tilde{R} \subset \tilde{R}' \text{ is a proper embedding} \]

\[ \tilde{G} \subset \tilde{G}' \text{ is a proper embedding} \]

and that we have a commutative diagram of proper embeddings:

\[
\begin{array}{ccc}
\tilde{R}' & \rightarrow & \tilde{R}_1' \\
\downarrow & & \downarrow \\
\tilde{R}_2' & \rightarrow & \tilde{R}_3'
\end{array}
\]

where:

\[ \tilde{R}_1' = \lambda ((R \setminus S) \times S^2 \times \{I\}) \]

\[ \tilde{R}_2' = \lambda ((R \setminus S) \times \{I\} \times S^2). \]

In addition, we observe that:

\[ \tilde{G}_1' \subset \tilde{R}_1' \]

\[ \tilde{R} = \tilde{R}_1' \cap \tilde{R}_2'. \]

Identification of oriented tangent spaces

From the map \( f \), we conclude that if \( p \in R \), then:

\[ T_p(R \setminus S') = T_p(R \setminus S) \oplus S \oplus S. \]

Note: Considering the given trivialization of \( T((R')^\#) \) as \( T((S^2)^{S^2^+}) \) and the associated Riemannian metric, this splitting can be considered as a natural orthogonal splitting.

Recall that our conventions on orientations stipulate that the following exact sequences of oriented vector spaces be compatible:

\[
\begin{array}{cccccc}
1 & \rightarrow & T_p(R \setminus S) & \rightarrow & T_p(R^\#) & \rightarrow & T_{3(p)}(R_2) & \rightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & T_p(R \setminus S') & \rightarrow & T_p((R')^\#) & \rightarrow & T_{3'(p)}(R_2') & \rightarrow & 1.
\end{array}
\]
With respect to the given basis, this diagram may be identified with:

\[
\begin{array}{c}
1 \rightarrow T_p(R/S) \rightarrow T_p((S^3)^\mathbb{R}) \rightarrow T_C(S^3) \rightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \text{id} \\
1 \rightarrow T_p(R'/S') \rightarrow T_p((S^3)^{2+2}) \rightarrow T_C(S^3) \rightarrow 1
\end{array}
\]

where \( c = \beta(\rho) = \beta'(\rho) \).

By the choice of orientations on \( \mathbb{R}^3 \) and \( (R')^\mathbb{R} \) given in part (i), we have the following equality of oriented vector spaces:

\[ T_p((R')^\mathbb{R}) = T_p(\mathbb{R}^3) \oplus S \oplus S \quad \text{(oriented)}. \]

From the definition of compatibility, we deduce:

\[ T_p(R'/S') = T_p(R/S) \oplus S \oplus S \quad \text{(oriented)}. \]

Using the same metric, we may split the tangent space to \( R/S \):

\[ T_p(R/S) = T_p(SO_3 \cdot \rho) \oplus \nu_p(SO_3 \cdot \rho) \]

\[ \nu_p(SO_3 \cdot \rho) = (T_p(SO_3 \cdot \rho))^\perp \oplus T_p(R/S). \]

Considering, the short exact sequences:

\[
\begin{array}{c}
1 \rightarrow T_p(SO_3 \cdot \rho) \rightarrow T_p(R/S) \rightarrow T_p(R) \rightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
1 \rightarrow T_p(SO_3 \cdot \rho) \rightarrow T_p(R'/S') \rightarrow T_p(R') \rightarrow 1
\end{array}
\]

we may make the following identifications:

\[ T_p(R) = \nu_p(SO_3 \cdot \rho) \]

\[ T_p(R') = \nu_p(SO_3 \cdot \rho) \oplus S \oplus S \]

\[ T_p(R') = T_p(R) \oplus S \oplus S . \]

Recall again that our conventions stipulate that the above sequences be compatibly oriented. That is, considering the above identifications:

\[ T_p(R/S) = T_p(R) @ T_p(SO_3 \cdot \rho) \quad \text{(oriented)} \]

\[ T_p(R'/S') = T_p(R') @ T_p(SO_3 \cdot \rho) \quad \text{(oriented)}. \]

From the previous observations, if we write:

\[ T_p(R) = \epsilon(T_p(R) @ S @ S) \quad \text{(oriented)} \]

then we obtain the equation:

\[ T_p(R) @ T_p(SO_3 \cdot \rho) @ S @ S = \epsilon(T_p(R) @ S @ S @ T_p(SO_3 \cdot \rho)). \]

Counting dimensions we see that \( \epsilon = 1 \):

3.) \[ T_p(R') = T_p(R) @ S @ S \quad \text{(oriented)}. \]

By the choice of orientations on \( Q_1 \) and \( Q_2 \) given in part (i), we have the equations:

\[ T_{h^\ast}((Q_1/S) = (h^\ast)_h(T_p(Q_1/S)) \quad \text{(oriented)} \]

\[ T_p(Q_1/S') = T_p(Q_1/S) @ S @ \{0\} \quad \text{(oriented)} \]

\[ T_p(Q_1/S') = T_p(Q_1/S) @ \{0\} @ S \quad \text{(oriented)}. \]

Our conventions stipulate the compatibility of:

\[
\begin{array}{c}
1 \rightarrow T_p(SO_3 \cdot \rho) \rightarrow T_p(Q_1/S) \rightarrow T_p(Q_1) \rightarrow 1 \\
\downarrow (h^\ast)_h \quad \downarrow (h^\ast)_h \quad \downarrow (h^\ast)_h \\
1 \rightarrow T_{h}(SO_3 \cdot \delta) \rightarrow T_{h}(Q_2/S) \rightarrow T_{h}(Q_2) \rightarrow 1
\end{array}
\]
where \( \phi = h^*(\rho) \).

We observe that the first column is orientation preserving:

\[
\begin{array}{ccc}
\text{h}^* : \text{SO}_3 \cdot \rho & \rightarrow & \text{SO}_3 \cdot \phi \\
\text{cpc}^{-1} & \rightarrow & \text{cpc}^{-1} \\
\end{array}
\]

Having already observed the same for the second column, we conclude that:

\[ T_\phi(\hat{Q}_2) = (\hat{h}^* \rho)(T_\phi(\hat{Q}_1)) \] (oriented).

Likewise, we must have compatibility of:

\[
\begin{array}{cccc}
1 & \rightarrow & T_\rho(\text{SO}_3 \cdot \rho) & \rightarrow & T_\rho(\hat{Q}_1 \setminus S) & \rightarrow & T_\rho(\hat{Q}_1) & \rightarrow & 1 \\
1 & \rightarrow & T_\rho(\text{SO}_3 \cdot \rho) & \rightarrow & T_\rho(\hat{Q}_1 \setminus S') & \rightarrow & T_\rho(\hat{Q}_1') & \rightarrow & 1. \\
\end{array}
\]

Following the argument for equation 3., write:

\[ T_\rho(\hat{Q}_1') = \varepsilon(T_\rho(\hat{Q}_1) \oplus S \oplus \{0\}) \] (oriented)

to obtain the equation:

\[
\varepsilon(T_\rho(\hat{Q}_1) \oplus S \oplus \{0\}) \oplus T_\rho(\text{SO}_3 \cdot \rho) \quad \text{(oriented)}
\]

\[ = T_\rho(\hat{Q}_1) \oplus T_\rho(\text{SO}_3 \cdot \rho) \oplus S \oplus \{0\}. \]

Counting dimensions we see that \( \varepsilon = -1 \):

\[ T_\rho(\hat{Q}_1') = -T_\rho(\hat{Q}_1) \oplus S \oplus \{0\} \] (oriented).

The same argument establishes the equation:

\[ T_\rho(\hat{Q}_1') = -T_\rho(\hat{Q}_1) \oplus S \oplus \{0\} \] (oriented).

5.) \[ T_\rho(\hat{Q}_1') = -T_\rho(\hat{Q}_1) \oplus \{0\} \oplus S \quad \text{(oriented)} \]

\[ T_\rho(\hat{Q}_1') = (\hat{h}^* \rho)(T_\rho(\hat{Q}_1')). \]

Construction of the isotopy

Choose a compactly supported ambient isotopy of \( \hat{R} \) which carries \( \hat{Q}_2 \) to \( \hat{Q}_2' \), where \( \hat{Q}_2 \) is transverse to \( \hat{Q}_1 \). By the isotopy extension theorem [Hi], extend this isotopy \( \hat{H} \) to an ambient isotopy of \( \hat{R} \). We assume that this isotopy respects the orthogonal splitting of the normal bundle of \( \hat{R} \):

\[
(\hat{H}_r) : T_\rho(R \setminus S) \oplus \{0\} \oplus S \rightarrow T_\rho(R \setminus S) \oplus \{0\} \oplus S \\
(x, \hat{Q}, y) \rightarrow (\hat{x}(x), \hat{Q}(y), \rho(y)).
\]

Note: The existence of such an isotopy follows readily from the proof of the isotopy extension theorem. Alternatively, one can construct the isotopy equivariantly in \( R \setminus S' \) using the product structure on the normal bundle of \( R \setminus S \).

Finally, extend to an isotopy of \( \hat{R}' \). Let \( \hat{Q}_2' = \hat{R}_1(\hat{Q}_2) \). The next observations follow immediately from the construction of \( \hat{R} \):

\[ \hat{Q}_3 = \hat{Q}_2 \circ \hat{R} \quad \hat{Q}_1 \circ \hat{Q}_2 = \hat{Q}_1' \circ \hat{Q}_2' = \hat{R} \]

\[ T_\rho(\hat{Q}_1') = T_\rho(\hat{Q}_1) \oplus \{0\} \oplus \rho \oplus \hat{R} \]

\[ T_\rho(\hat{Q}_2') = T_\rho(\hat{Q}_2) \oplus \{0\} \oplus \rho \oplus \hat{R}. \]

Since \( \hat{Q}_1 \) and \( \hat{Q}_2 \) are transverse, we conclude that:

\( \hat{Q}_1' \) and \( \hat{Q}_2' \) are transverse.
Moreover, by the previous results and the construction of the isotopy:
\[
\begin{align*}
T^\rho_\sigma(G_1) &= -T^\rho_\sigma(G_1) \ominus S_1 \ominus \{0\} \quad \text{(oriented)} \\
T^\rho_\sigma(G_2') &= -T^\rho_\sigma(G_2') \ominus \{0\} \ominus S_2 \quad \text{(oriented)}.
\end{align*}
\]

Let:
\[
\begin{align*}
s &= \text{sign}(\rho; \tilde{\eta}_1, \tilde{\eta}_2) \\
s' &= \text{sign}(\rho; \tilde{\eta}_1', \tilde{\eta}_2).
\end{align*}
\]

Then we have the equations:
\[
\begin{align*}
T^\rho_\sigma(R') &= T^\rho_\sigma(R) \ominus S_1 \ominus S_2 \quad \text{(oriented)} \\
T^\rho_\sigma(R) &= s(T^\rho_\sigma(G_1) \ominus T^\rho_\sigma(G_2)) \quad \text{(oriented)} \\
T^\rho_\sigma(R') &= s'(T^\rho_\sigma(G_1') \ominus T^\rho_\sigma(G_2')) \quad \text{(oriented)}.
\end{align*}
\]

From this, we derive the equations:
\[
s(T^\rho_\sigma(G_1) \ominus T^\rho_\sigma(G_2)) \ominus S_1 \ominus S_2 \quad \text{(oriented)}
\]
\[
= s'((-T^\rho_\sigma(G_1) \ominus S_1) \ominus (-T^\rho_\sigma(G_2) \ominus S_2)).
\]

Counting dimensions, we conclude that:
\[
s = s'(-1)^3(1-2).
\]

Since this holds for all \( \rho \in \tilde{\eta}_1 \cap \tilde{\eta}_2 \), we conclude from the previous observations that:
\[
\langle \tilde{\eta}_1, \tilde{\eta}_2 \rangle^R = (-1)^{s-1} \langle \tilde{\eta}_1, \tilde{\eta}_2 \rangle^R.
\]

From equations 1., 2. and 6., we conclude:
\[
\lambda(W', h') = \lambda(W, h).
\]

2. Casson's invariant for oriented homology 3-spheres

(a) Casson's invariant for oriented homology 3-spheres

If \( \mathbb{M}^3 \) is an oriented homology 3-sphere, we define:
\[
\lambda(\mathbb{M}^3) = \lambda(\mathbb{M}^3; W_1, W_2)
\]
where \((W_1, W_2)\) is an Heegard dec. of \( \mathbb{M}^3 \).

From the results of section 1(c), this is well-defined.

If \( \mathbb{M}^3 \) and \( \mathbb{N}^3 \) are oriented homology 3-spheres and \( h : \mathbb{M}^3 \to \mathbb{N}^3 \) is an orientation preserving homeomorphism, then \( h \) carries any given Heegard decomposition of \( \mathbb{M}^3 (W_1, W_2) \) to an Heegard decomposition of \( \mathbb{N}^3 (h(W_1), h(W_2)). \) It is evident from our conventions that:
\[
\lambda(\mathbb{M}^3; W_1, W_2) = \lambda(\mathbb{N}^3; h(W_1), h(W_2)).
\]

Hence, we obtain a topological invariant:

Proposition 2.1: Casson's invariant for oriented homology 3-spheres \( \lambda \) is an homeomorphism invariant.

The obvious question at this point is whether Casson's invariant is computable. It is well-known that every homology 3-sphere \( \mathbb{M}^3 \) is obtained from \( S^3 \) by a sequence of \( \# 1 \) surgeries on knots in homology 3-spheres.

Hence, one approach to this question is to study the change in Casson's invariant when we perform \( \# 1 \) surgery on a knot in \( \mathbb{M}^3 \). It turns out that there is a very simple formula for this change (as we shall see). In the next chapter we begin the study of this "differential".
CHAPTER V: CASSON'S INVARIANT FOR KNOTS IN HOMOLOGY 3-SPHERES

1. Dehn surgery on knots in homology spheres
   1a. Dehn surgery on knots in homology spheres

   Let $M^3$ be an homology 3-sphere.
   Let $K$ be a knot in $M^3$.
   Let $N(K)$ be a closed regular neighborhood of $K$.
   Let $T^2$ be the boundary of $N(K)$, $\partial N(K)$, so that $T^2$ is a torus.
   Let $M^3(K)$ be $M^3 \setminus \text{interior}(N(K))$.

   Now:
   $$H_1(M^3(K)) \cong H_1(N(K)) = \mathbb{Z}$$
   $$H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}.$$

   Consider the inclusions:
   $$i : T^2 \rightarrow N(K)$$
   $$j : T^2 \rightarrow N(K)$$

   and the associated surjections:
   $$i_* : H_1(T^2) \longrightarrow H_1(M^3(K))$$
   $$j_* : H_1(T^2) \longrightarrow H_1(N(K)).$$

   Let $\mu$ be a generator of the kernel of $i_*$. Let $\lambda$ be a generator of the kernel of $j_*$. Since $\mu$ and $\lambda$ are primitive elements, we may represent them by nontrivial simple closed curves on $T^2$ which we also denote as $\mu$ and $\lambda$. Since $\mu$ and $\lambda$ form a basis for $H_1(T^2)$, we may assume that they intersect transversely in precisely one point. Each of these curves is well defined up to isotopy and change in orientation.

   Let $M^3$ be oriented. The orientation on $M^3$ restricts to an orientation on $M^3(K)$. This orientation on $M^3(K)$ induces a well defined orientation on $T^2$ by the convention that "the orientation on $M^3(K)$ is the orientation on $T^2$ plus the inward pointing normal vector to $T^2".$ (We assume that all orientations are chosen subject to this convention.) We choose the orientations on $\mu$ and $\lambda$ so that "the orientation on $T^2$ is equal to the orientation on $\mu$ plus the orientation on $\lambda".$ Subject to these assumptions, the pair of curves $(\mu, \lambda)$ is well defined up to ambient isotopy of $T^2$ and simultaneous change of orientation of $\mu$ and $\lambda$. The pair is called the standard meridian-longitude pair for $K$.

   ![Figure 14](image)

   **Figure 14**

   Any simple closed curve $\gamma$ on $T^2$ is uniquely determined by its homology class (which is well defined up to a change in sign):
   $$\gamma = p\mu + q\lambda \quad \langle p, q \rangle = 1.$$

   Moreover, the relative primeness of $p$ and $q$ ensure that $p\mu + q\lambda$ is
represented by a simple closed curve. Hence, for each fraction $p/q$ we obtain a well-defined (up to isotopy) homotopically nontrivial simple closed curve on $T^3$, and vice-versa. (We allow that $(p,q) = (\pm 1,0)$ or $p/q = \infty$.)

$M^3(K,p/q)$ is $p/q$ surgery on $(M^3, K)$, (or just $p/q$ surgery on $K$).

This is defined as follows:

$$M^3(K, p/q) = M^3(K) \cup_{\sigma} (S^1 \times D^2)$$

$\sigma : S^1 \times \partial D^2 \rightarrow T^3$

$(1) \times \partial D^2 \rightarrow \gamma$

($M^3(K, p/q)$ does not depend on the choice of $\sigma$.)

Now $H_1(M^3(K, p/q)) = \mathbb{Z}^p$ (assuming $p > 0$). Hence $M^3(K, p/q)$ is an homology 3-sphere precisely when $p = 1$.

$$K_n = M^3(K, 1/n) \text{ is } 1/n \text{ Dehn surgery on } K.$$  

Note: $K_n = M^3(K, n) = M^3$.

(b) Preferred Heegaard decompositions for $K_n$

In this section, we construct Heegaard decompositions which are compatible with surgery. These will be useful in comparing the Casson invariants for $K_n$ and $K_{n+1}$.

Lemma 1.1: There exists an Heegaard decomposition for $M^3(W_1, W_2)$ such that $K$ is a separating curve on $\partial W_1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15}
\caption{Figure 15}
\end{figure}

Proof: Choose a Seifert surface $F^*$ for $K$. Thicken $F^*$ to $F^* \times I \subset M^3$.
(Here, $I = [0,1]$ and $F^*$ is identified with $F^* \times 1$.)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure16}
\caption{Figure 16}
\end{figure}

Attach 1-handles (from $M^3 \setminus (F^* \times I)$) to $F^* \times [0]$ to obtain a new handlebody $W_1$ such that $M^3 \setminus W_1$ is an handlebody $W_2$.

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By an ambient isotopy, we may assume \( K = 3F \times (1/2) \). 

Now choose an Heegaard decomposition of \( M^3 \) as in Lemma 1.1 (or Lemma 1.2):

\[ 2W = F' \cup (3F' \times I) \cup F' \]

Let:

\[ \lambda = 3F \times I, \quad \lambda = 3F' \times \{0\} \]

By an ambient isotopy, we assume that \( K = 3F' \times (1/2) \).
Note: $K$ is homologically trivial in $\mathcal{W}_1$. It is not, however, homotopically trivial in $\mathcal{W}_1$, unless $F'$ is a disc, and, hence, $K$ is a trivial knot.

Let $W$ denote $\mathcal{W}_1$ and identify the given Heegard decomposition of $M^3$ with an Heegard model $(W,h)$. Let $\tau$ denote the Dehn twist about $K$ which we take to be supported on $A$.

**Lemma 1.3:** $(W,h\circ \tau)$ is an Heegard model for $M^3(K,1/n)$.

**Proof:** Let $B$ be $h(A)$ in $\mathcal{W}$. Then $M^3(K)$ is homeomorphic to the quotient space:

$$\mathcal{W} \times \{1\} \cup (-\mathcal{W} \times \{2\})$$

where:

$$h' = h|_{\mathcal{W} \setminus A} : \mathcal{W} \setminus A \to W \setminus B$$

Identifying $M^3(K)$ with this quotient, we observe that:

$$T^3 = A \cup B$$

$$\lambda = 3F \times \{0\} = \text{longitude}.$$  

Let $\alpha$ be an arc in $A$ from $3F \times \{0\}$ to $3F \times \{1\}$. Let $a = h(\alpha)$. Likewise, we observe that:

$$\mu = a \cup \beta = \text{meridian}.$$  

Since $\tau$ is supported on $A$ we may consider $\tau$ as a Dehn twist on $T^3$.

Under our conventions on orientations, we have:

$$\tau^B(\mu) = \mu + n\lambda = \tau^B(a) \cup \beta.$$  

(Remember, in general, $\tau_0(\text{d}) = \text{d} + (\text{c} \cdot \text{d}) \mu \nu$. Since $(\lambda,\mu) = 1$, we obtain a change of sign.)

![Figure 20](image)

Let $\alpha' = \tau^B(\alpha)$, so that:

$$\beta = h \circ \tau^B(\alpha')$$

$$\mu' = a' \cup \beta = \mu + n\lambda.$$  

In order to obtain $M^3(K,1/n)$ we wish to attach a disc to $\mu$ and cap off. From the preceding observation and the same considerations which allow us to identify $M^3(K)$ as a "prequotient" of $M^3$, we conclude, as desired, that:

$$M^3(K,1/n) \cong \left(\mathcal{W},h \circ \tau^B\right)\left(\mathcal{W} \setminus A\right).$$

2. Casson's invariant for framed knots

(a) Casson's invariant for framed knots

A framed knot is a pair $(K,n)$ where $K$ is a knot and $n$ is an integer. If $K$ is a knot in an homology 3-sphere, as above, we define:

$$\lambda'(K,n) = \lambda(K_{n+1}) - \lambda(K_n).$$
The canonical isotopy

If we choose Heegaard models for the Dehn surgeries on \( K \) as in the previous section, we compute, as in section IV.1(c), that:

\[
\lambda(K_n) = \frac{(-1)^g \langle \hat{Q}_1, (\eta^n) \hat{Q}_1 \rangle_R}{2Q_1, (\eta^n) Q_2_R^*} \cdot S
\]

where \( g \) is genus(\( F \)) and \( 2g \) is genus(\( \mathcal{M} \)). That is:

\[
\lambda(K_n) = \frac{\langle \hat{Q}_1, (\eta^n) \hat{Q}_2 \rangle_R}{2Q_1, (\eta^n) Q_2_R^*} \cdot S
\]

where \( Q_2 \) is \( h(Q) \).

Since \( K \) is a separating curve, \( \tau \) acts trivially on the homology of \( \mathcal{M} \). Following the proof of Proposition III.1.1, part (a), we conclude that:

\[
\langle Q_1, (\eta^n) Q_2 \rangle_R^* = Q_1, Q_2_R^*.
\]

Assume, as in section IV.1(b), that \( \langle Q_1, Q_2 \rangle_R^* \) is 1. With this assumption, we have:

\[
\lambda(K_n) = \left( \langle \hat{Q}_1, (\eta^n) \hat{Q}_2 \rangle_R/2 \right)^* \cdot S.
\]

Allowing thru our conventions on orientations, we see that \( \tau(\mathcal{R}) \) is \( \mathcal{R} \) (as an oriented manifold) and, hence, that:

\[
\lambda(K_{n+1}) - \lambda(K_n) = \frac{1}{2} \langle \hat{Q}_1, (\eta^{n+1}) \hat{Q}_2 \rangle_R - \langle \hat{Q}_1, (\eta^n) \hat{Q}_2 \rangle_R \cdot S
\]

\[
= \frac{1}{2} \langle \hat{Q}_1, (\eta^{n+1}) \hat{Q}_2 \rangle_R - \langle \hat{Q}_1, (\eta^n) \hat{Q}_2 \rangle_R \cdot S.
\]

We wish to study the difference \( \check{Q}_1 = \hat{Q}_1 \).

Note: It should be emphasized here that \( \check{Q}_1 \) is not a cycle in the classical sense, since it is not compact.

We may identify \( \mathcal{R} \) in the following manner:

\[
(R^\ast) = R(\mathcal{R}(F')) \equiv (S^g) \quad g_1 = \text{genus}(F')
\]

\[
(R^\ast) = R(\mathcal{R}(F'')) \equiv (S^g) \quad g_2 = \text{genus}(F'')
\]

\[
R = R(\mathcal{R}(\mathcal{M})). \subset (R^\ast) = (R^\ast)^*
\]

\[
R = \{ (\rho', \rho^\ast) \mid \rho' = \rho^\ast \}
\]

In these coordinates, \( \tau^\ast \) has a simple form:

\[
\tau^\ast : R \longrightarrow R
\]

\[
(\rho', \rho^\ast) \longmapsto (\rho', \rho^- \cdot \rho')
\]

Note: We choose the base point for \( \mathcal{M} \) in \( F' \). As above, \( u = u\rho = u^{-1} \).

\[
R'_+ = \{ \rho' \in (R^\ast) \mid \rho' = -1 \}
\]

\[
R'_- = \{ \rho^\ast \in (R^\ast) \mid \rho^- = -1 \}
\]
Let \( R_\times(\mathbf{K}) \) be \( R_\times \times R_\times \) in \( R \). Since reducible representations are abelian and \( \pi \) is homologous to zero, we readily deduce that:

\[
R_\times(\mathbf{K}) \cap S = \emptyset
\]

\[
\tau^\# |_{R_\times(\mathbf{K})} = \text{identity} \quad \tau^\# |_S = \text{identity}
\]

Let:

\[
P_n = (b r^{n+1})^\# Q_1.
\]

Since \( K_{n+1} \) is a homology sphere, we know that:

\[
Q_1 \notin P_n \text{ at 1}
\]

\[
Q_1 \cap P_n \cap S = \{1\}
\]

From the observation concerning \( \tau^\# \), we conclude:

\[
\tau^\#(Q_1) \cap S = Q_1 \cap S
\]

\[
\tau^\#(Q_1) \cap R_\times(\mathbf{K}) = Q_1 \cap R_\times(\mathbf{K})
\]

In the complement of \( R_\times(\mathbf{K}) \), \( \tau^\# \) is given by conjugation of \( \tau^\times \) by an element of \( S^3 \setminus \{-1\} \). Hence, we can employ the natural contraction of \( S^3 \setminus \{-1\} \) to \( \{1\} \) which arises from the exponential map, (see sections 3.1(b) and (c)), to construct an isotopy in the complement of \( R_\times(\mathbf{K}) \) which carries \( Q_1 \) onto \( \tau^\#(Q_1) \):

\[
H : (R \setminus R_\times(\mathbf{K})) \times I \longrightarrow R \setminus R_\times(\mathbf{K})
\]

\[
((\rho', \rho^*), t) \longrightarrow (\rho', (2\rho')^t \cdot \rho^*)
\]

**Figure 21**

**Note:** In Figure 21, the shaded region indicates the trace of \( Q_1 \) (away from \( R_\times(\mathbf{K}) \)) under the isotopy \( H \). It should be observed that \( H \) does not extend to \( R_\times(\mathbf{K}) \).
Clearly \( II_{S \times I} \) is identity.

We recall from the proof of Proposition 1.3.3 that the tangent space at 1 to 
\( z \) is spanned by curves of cyclic representations (i.e. the Zariski tangent 
pace at 1 to \( S \) is \( T_1(RP^1) \)). Moreover, for any \( t \in I \):

\[
T_{(t,1)}(\{(R \times R_{-}\langle K \rangle) \times I \}) = T_{1}(R \times R_{-}\langle K \rangle) \oplus T_{t}(I).
\]

Note: In a small neighborhood of 1, the isotopy extends in an obvious way 
over an open interval containing 1. It is with this consideration that we speak 
of \( T_0(I) \) and \( T_1(I) \).

From the previous observations we conclude that:

\[
E_{(t,1)}(R) = \Pi_{1} : T_{(t,1)}((R \times R_{-}\langle K \rangle) \times I) \rightarrow T_{1}(R \times R_{-}\langle K \rangle)
\]

where \( \Pi_{1} \) is projection onto the first factor. It follows by standard 
techniques that there exists an open neighborhood of 1 in \( R \times U_n \) such 
that:

\[
\Pi((Q \times U_n) \times I) \cap P_n = \{1\}.
\]

We are now able to prove an analogous result for the trace of \( Q \) near \( S \):

**Lemma 2.1:** There exists a neighborhood of \( Q \cap S \) in \( Q \), \( V_n \), such that:

\[
\Pi(V_n \times I) \cap P_n = \{1\}.
\]

**Proof:** From the previous observations, it follows that:

\[
((Q \cap S) \cap U) \cap (\langle P \cap S \rangle \cap U) = 0,
\]

\[
(Q \cap S) \cap U \text{ and } (\langle P \cap S \rangle \cap U) \text{ are compact.}
\]

- Moreover, \( H_{L} \) fixes both sets for all \( L \).

- Hence, we may choose a neighborhood of \( (Q \cap S) \cap U_n \) in \( Q \), \( W_n \), such 
  that:

\[
\Pi(W_n \times I) \cap P_n = \{1\}.
\]

- Finally, let \( V_n \) be \((Q \cap U_n) \cup W_n \).

**Figure 23**
Note: From the conclusion, we see that we may choose $V_n$ to be $SO_3$ invariant. Moreover, we can assume that $Q_1 \setminus V_n$ is a manifold with boundary. Clearly, $\tilde{R}$ is $SO_3$-equivariant. Hence, we have a canonical isotopy $\tilde{H}$ defined on $\tilde{R}$ in the complement of $\tilde{R}_-(K)$ and carrying $\tilde{Q}_1$ onto $\tilde{r}(\tilde{Q}_1)$. Moreover:

$$\tilde{H}(\tilde{Q}_1 \setminus V_n) = \tilde{H}(\tilde{Q}_1 \setminus V_n).$$

Note: Recall that $\tilde{R} = (R \setminus S)/SO_3$. All quotient spaces, therefore, are contained in this space, (by definition). We will, however, for convenience, speak of $\tilde{R}$ and consider that:

$$\tilde{R} = R/SO_3, \tilde{R} = R/SO_3,$$

$$\tilde{R} \cap \tilde{S} = \varnothing, \tilde{R} \cup \tilde{S} = R/SO_3.$$

(c) The difference cycle $\delta$

Although $\tilde{R}$ is not a compactly supported isotopy, the results of the previous section allow us to employ $\tilde{R}$ to define a compact cycle $\delta$ which "carries the intersection of $\tilde{r}(\tilde{Q}_1) - \tilde{Q}_1$ with $\tilde{R}_n". As we shall see, this difference cycle is independent of $n$ (as a genuine cycle in $\tilde{R}$) and its intersection with $\tilde{R}_n$ is independent of $n$. This will establish Casson's knot invariant.

Let $\delta'$ be the compact cycle:

$$\delta' = \tilde{r}(\tilde{Q}_1 \setminus \tilde{V}_n) - \tilde{H}(\tilde{Y}_n \setminus I) - (\tilde{Q}_1 \setminus \tilde{V}_n).$$

Figure 24

By the discussion in the previous section, we conclude that:

$$\langle \tilde{r}(\tilde{Q}_1), \tilde{P}_n \rangle_R - \langle \tilde{Q}_1, \tilde{P}_n \rangle_R = \langle \delta', \tilde{P}_n \rangle_R.$$
Note: $\delta'$ is a compact cycle in $\hat{N}$.

We assume that $\hat{V}_n$ is disjoint from $\hat{N}(K)$. Let $\hat{N}$ be a compact manifold neighborhood of $\hat{N} \cap \hat{N}(K)$ in $\hat{N}$, with $\hat{N} \cap \hat{V}_n = \emptyset$. Let:

$$\hat{N} = \hat{N} \setminus (\hat{V}_n \cup \text{interior}(\hat{N}))$$

be chosen so that $\hat{N}$ is a manifold. Let

$$\beta = \tau(\hat{N}) - \hat{N}(3\hat{N} \times I) - \hat{N}$$

Since $\beta$ is a compact boundary in $\hat{N}$:

$$\langle \beta, \hat{P} \rangle_{\hat{N}} = 0$$

Finally, we define the difference cycle $\delta$:

$$\delta = \tau(\hat{N}) - \hat{N}(3\hat{N} \times I) - \hat{N}$$

Then $\delta = \delta' - \beta$, we conclude that:

$$\langle \tau(\hat{N}), \hat{P} \rangle_{\hat{N}} - \langle \delta', \hat{P} \rangle_{\hat{N}} = \langle \delta, \hat{P} \rangle_{\hat{N}}$$

Clearly, $\delta$ is independent of the choice of $\hat{N}$, and independent of $n$.

1) Independence of framing

From the previous sections, we conclude that:

$$\lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \delta, \hat{P} \rangle_{\hat{N}}\cdot S$$

Applying the fact that $\tau(\hat{N})$ is $\hat{N}$ (as an oriented manifold), we have:

$$\lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \tau(\hat{N}) \delta, \hat{P} \rangle_{\hat{N}}\cdot S$$

It suffices to show, therefore, that $\tau(\hat{N})$ is homologous to $\delta$ in $\hat{N}$. From the previous section, we see that we can assume that $\delta$ is carried by a regular neighborhood of $\hat{N}(K)$ and, hence, that $\delta$ is in the image of the natural map:

$$i_* : \pi_h(\hat{N}(K)) \longrightarrow \pi_h(\hat{N})$$

We have already seen that $\tau$ acts trivially on $\hat{N}(K)$; (indeed, $\tau^a$ acts trivially on $\hat{N}(K)$). This establishes the result. In summary, we have shown that:

$$\lambda'(K, n) = \lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \delta, \hat{P} \rangle_{\hat{N}}\cdot S$$

for any Heegaard decomposition as in Lemma 1.1. Hence, $\lambda'(K, n)$ is independent of $n$.

3. Casson's invariant for knots

(a) Casson's invariant for knots

Given a knot $K$ in an homology $3$-sphere, we define Casson's invariant by:

$$\lambda'(K) = \lambda(K_{n+1}) - \lambda(K_n) \quad n \in \mathbb{Z}$$

By the previous section, this is a well defined topological invariant. Moreover:

$$\lambda'(K) = -\frac{1}{2} \langle \delta, \hat{P} \rangle_{\hat{N}}\cdot S$$

Note: In order to compute this invariant, we investigate the difference cycle $\delta$. 

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The difference cycle \( \delta \)

(a) Identification of representation spaces

Using an Ito-Geometric model as in Lemmas 1.2 and 1.3, we may identify representation spaces as follows:

\[ R'(\mathbb{R})^* = R(\Pi_1(\mathbb{R})) \cong (S^3)^* \quad \text{and} \quad \delta_e = \text{genus}(R') \]

\[ R^* = R(\Pi_1(\mathbb{R} \setminus \{0\})) = (R')^* \times (R')^* \]

\[ \Pi = R(\Pi_1(\mathbb{R})) \subset R^* \]

\[ H = \{(r', r^*) \mid 2r' = 3r^*\} \]

\[ H = \{(r', r^*) \mid r'(\lambda) = r^*(\lambda)\} \]

\[ Q_1 = H \subset (R')^* \times (R')^* = R^* \quad \text{(and diagonal)} \]

\[ Q_2 = (R')^* \]

\[ S' = \text{reducible representations in } (R')^* \]

\[ S = \text{reducible representations in } R^* \]

\[ \Pi_1(K) = R' \times R' \]

\[ Q_1 \cap R_1(K) = \{(r', r^*) \mid 2r' = -I\} \]

\[ N_\varepsilon = \text{regular neighborhood of } Q_1 \cap R_1(K) \text{ in } Q_1 \]

\[ N_\varepsilon = \{(r', r^*) \mid H - \varepsilon \leq \text{arg}(3r') \leq H\} \]

\[ \tau : R^* \longrightarrow R^* \]

\[ (r', r^*) \longmapsto (r', 3r^\cdot r^*) \]

\[ H : (R \setminus R_1(K)) \times I \longrightarrow R \setminus R_1(K) \]

\[ (r', (\varepsilon, r^*)) \longmapsto (r', (3r^\cdot r^*)^\varepsilon \cdot r^*) \]

\[ \delta_e = \tau(N_\varepsilon) - \tau(3N_\varepsilon \times I) - \tilde{N}_e \]

Note: We shall show that \( \delta_e \) converges to a cycle in \( \tilde{N}_e \).

(b) An equivariant trivialization of \( N_\varepsilon \)

We may identify \( Q_1 \) with \( (R')^* \) via the diagonal map:

\[ d : (R')^* \longrightarrow R^* \]

\[ r' \longmapsto (r', r') \]

\[ d : (R')^* \longrightarrow Q_1 \]

With this identification, we may write:

\[ Q_1 = R_1(K) \cong E_1 \]

\[ N_\varepsilon = \{r' \mid H - \varepsilon \leq \text{arg}(3r') \leq H\} \]

Let:

\[ N_- = \{r' \mid \text{trace}(3r') = 0\} \]

\[ N_- = \{r' \mid H/2 \leq \text{arg}(3r') \leq H\} \]

\[ \dim \rightarrow \frac{\Pi}{T/2} \]

\[ (R')^* \]

\[ N_- \]

\[ S^3 \]

Figure 25
By the results of section III.1(b), we may choose a trivialization:

\[ w : R' \times D^3_\varepsilon \rightarrow N_- \]
\[ w(p',\alpha) = \alpha \quad w(p',-1) = p' \]

If \( \varepsilon \neq \pi/2 \), we have \( N_-(\varepsilon) \subset N_- \) and the trivialization restricts to:

\[ w : R'_\varepsilon \times D^3_\varepsilon \rightarrow N_-(\varepsilon) \]

\( w \), for sufficiently small \( \varepsilon \), the exponential map on \((R')^\varepsilon\) provides a projection which is equivariant with respect to the \( \text{SO}_3 \) action:

\[ \Pi : N_-(\varepsilon) \rightarrow R'_\varepsilon \]

After projecting the normal bundle of \( R'_\varepsilon \) in \((R')^\varepsilon\), we obtain a homeomorphism:

\[ v : N_-(\varepsilon) \rightarrow R'_\varepsilon \times D^3_\varepsilon \]
\[ \rho' \mapsto (\Pi(p'),\varepsilon p') \]

We may assume that (for sufficiently small \( \varepsilon \)):

\[ w = \text{identity} \]

\( w \) is an equivariant trivialization.

\section*{Collapse of \( \delta \) to a cycle in \( \hat{\mathbb{R}}_\varepsilon(K) \)

As we have already pointed out in section 2(c), \( \delta \) is carried by \( \hat{\mathbb{R}}_\varepsilon(K) \) or more precisely by a regular neighborhood of \( \hat{\mathbb{R}}_\varepsilon(K) \). In this section, we collapse \( \delta \) into \( \hat{\mathbb{R}}_\varepsilon(K) \) and obtain an alternative description of \( \delta \). Let:

\[ \overline{D}^3_\varepsilon \delta = \{ A \in S^3 : 0 < \arg(A) < \varepsilon \} \]
\[ S^3_+ = \{ A \in S^3 : \arg(A) = \varepsilon \} \]
\[ S^3_- = \{ A \in S^3 : \arg(A) = \pi - \varepsilon \} \]
\[ D^2_\varepsilon \delta = \{ A \in S^3 : \Pi - \varepsilon < \arg(A) < \pi \} \]

\[ \text{Figure 26} \]

We shall require a trivialization of \( A_\varepsilon \) obtained via the exponential map in projecting to the "bottom end":

\[ \text{Figure 27} \]
\( \mu_e : A_e \longrightarrow S^2(\varepsilon) \times [0,1] \)
\[
\times \longmapsto (x, t_e)
\]

where:
\[
\varepsilon = \arg(x)
\]
\[
t_e = (\varepsilon - \varepsilon)^2 /
\]
\[
t_e = (\varepsilon + \varepsilon - \pi)/(2\varepsilon - \pi).
\]

We observe that this decomposition of \( S^2 \) is preserved under the natural involution:
\[
i : (S^2, B^2_+(\varepsilon), A_+^2(\varepsilon), A_0^2(\varepsilon), B^2_-(\varepsilon)) \longrightarrow (S^2, B^2_-(\varepsilon), B^2_+(\varepsilon), A_0^2(\varepsilon), B^2_+(\varepsilon)).
\]

The trivialization of \( A_e \) and the involution \( i \) are compatible in that:
\[
\mu_e(x) = (x, t_e) \text{ iff } \mu_e(x^{-1}) = (x, 1 - t_e).
\]

Clearly, we may write:
\[
B_+^2 \times S^2 = (B_+^2 \times B^2_+(\varepsilon)) \cup (B_+^2 \times A_0) \cup (B_+^2 \times B^2_-(\varepsilon)).
\]

The trivialization above allows us to realize \( \delta_e \) via a map from \( \mathbb{R}_+^2 \times S^2 \).

Observe that as a cycle:
\[
\delta(N_e \times I) = \tilde{N}_e \times [1] + (\tilde{N}_e \times [1]) + \tilde{N}_e \times [0].
\]

We have also previously remarked that \( \tilde{R} \) does not extend across \( R_-(K) \).

However, in the complement of \( R_-(K) \), we know that:
\[
R_0 = i_0 = \text{inclusion} \quad R_1 = \tau_1
\]

Hence, we may extend \( \tilde{R} \) continuously to:
\[
\tilde{R} : (\mathbb{R} \times \{0\}) \cup ((\mathbb{R} \setminus \mathbb{R}(K)) \times I) \cup (R \times \{1\}) \longrightarrow \mathbb{R}.
\]

With this extension in mind, we may write:
\[
\delta_e = \tilde{R}(\delta(N_e \times I)).
\]

For the purpose of the next lemma, we define the following actions of \( SO_3 \) on \( \mathbb{R}_-^2 \times S^2 \) and on \( N_e \times I \):
\[
SO_3 \times (\mathbb{R}_-^2 \times S^2) \longrightarrow \mathbb{R}_-^2 \times S^2
\]
\[
(C, (\alpha, \beta)) \longrightarrow (C \cdot \alpha', C \cdot \beta', C \cdot \alpha, C \cdot \beta)
\]
\[
SO_3 \times (N_e \times I) \longrightarrow N_e \times I
\]
\[
(C, (\alpha, \beta)) \longrightarrow (C \cdot \alpha', C \cdot \beta)
\]

**Lemma 4.1:** There exists a natural \( SO_3 \)-equivariant map:
\[
\delta_e : \mathbb{R}_-^2 \times S^2 \longrightarrow \beta(N_e \times I)
\]
such that:
\[
\delta_e = \tilde{R}(\delta_e(\mathbb{R}_-^2 \times S^2)).
\]

**Note:** From the previous observation:
\[
\delta_e = \tilde{R}(\delta_e(\mathbb{R}_-^2 \times S^2)).
\]

All we need to do is to construct \( \delta_e \) as a degree one homeomorphism. Actually, we shall ensure that \( \delta_e \) converges as \( \varepsilon \) tends to zero. In this way, we shall realize \( \delta_e \) as a class in \( \mathbb{R}_-(K) \).

**Proof:** We construct \( \delta_e \) "in pieces."
Let:

\[ f_\varepsilon = Hg_\varepsilon : R_\varepsilon \times S^2 \to R \]

Then:

\[ g_\varepsilon = \overline{f_\varepsilon (R_\varepsilon \times S^2)} \]

Our next goal is to study the limiting behavior of \( f_\varepsilon \) as \( \varepsilon \) tends to zero. With this in mind, we define:

\[ f_- : R_- \times S^2 \to R_-(N) \]

\[ (\rho', u) \mapsto (\rho', u^{-1} \cdot \rho') \]

Clearly, \( f_- \) is \( SO_3 \)-equivariant.

Lemma 4.2: \( \lim_{\varepsilon \to 0} f_\varepsilon = f_- \)

Proof: We check the limit "in pieces". Let \((\rho', x) \in R_- \times S^2\).

Case (1): \( x = -I \)

\[ f_\varepsilon (\rho', x) = f_\varepsilon (\rho', -I) = i_{\varepsilon} (g_\varepsilon (\rho', -I)) \]

\[ = i_{\varepsilon} (w(\rho', -I), w(\rho', -I), 0) \]

\[ = (\rho', \rho') \]

\[ f_-(\rho', x) = (\rho', (-I)^{-1} \cdot \rho') = (\rho', \rho') \]

Hence:

\[ \lim_{\varepsilon \to 0} f_\varepsilon (\rho', x) = f_-(\rho', x) \]
\[ f_\varepsilon(p', x) = \tau \cdot g_\varepsilon(p', 1) \]
\[ = \tau(w(p', 1^{-1}), w(p', 1^{-1})) \]
\[ = \tau(p', p') \]
\[ = (p', (2p')^{-1}) \]
\[ = (p', p') \]
\[ f^\varepsilon(p', x) = (p', 1^{-1} \cdot p') = (p', p') . \]

Hence:
\[ \lim_{\varepsilon \to 0} f^\varepsilon(p', x) = f^\varepsilon(p', x) . \]

Case (3): \( x \neq 1 \)

Choose \( \delta \) sufficiently small such that \( x \in A_\varepsilon \) for all \( \varepsilon < \delta \). Then, if \( \varepsilon < \delta \):
\[ f^\varepsilon(p', x) = h^\varepsilon(p', x) \]
\[ = H(w(p', x_\varepsilon), w(p', x_\varepsilon), t_\varepsilon) . \]

Let:
\[ p_\varepsilon = w(p', x_\varepsilon) \quad \text{so} \quad q_\varepsilon = x_\varepsilon . \]

Then:
\[ f^\varepsilon(p', x) = (p_\varepsilon, (2p_\varepsilon)^{-1} \cdot p_\varepsilon) \]
\[ = (p_\varepsilon, (2p_\varepsilon)^{-1} \cdot p_\varepsilon) . \]

Clearly:
\[ \lim_{\varepsilon \to 0} x_\varepsilon = -I . \]

Hence:
\[ \lim_{\varepsilon \to 0} p_\varepsilon = w(p', 1^{-1}) = p' . \]

Claim: \( \lim_{\varepsilon \to 0} (x_\varepsilon)^{t_\varepsilon} = -I . \)

Note: We recall that the natural contraction on \( S^3 \setminus \{1\} \) does not extend to \( S^3 \).

Proof of Claim:
Recall that \( x_\varepsilon = x_{\varepsilon}^S \) and \( \arg(x_{\varepsilon}^S) = \pi - \varepsilon \). Hence:
\[ \arg(x_{\varepsilon}^S) = \pi - \varepsilon . \]

Let:
\[ \theta = \arg(x), \quad \text{so that} \quad s_\varepsilon = \frac{\pi - \varepsilon}{\theta} . \]

Now, by definition, \( t_\varepsilon = (\varepsilon + \theta - \pi)/(2\varepsilon - \pi) \). Now:
\[ (x_\varepsilon)^{t_\varepsilon} = x_{\varepsilon}^{s_\varepsilon} . \]

Moreover:
\[ a_{\varepsilon}^{t_\varepsilon} = \frac{(\pi - \varepsilon) \cdot (\varepsilon + \theta - \pi)}{\theta \cdot (2\varepsilon - \pi)} . \]

so that:
\[ \lim_{\varepsilon \to 0} a_{\varepsilon}^{t_\varepsilon} = \frac{\pi - \theta}{\theta} . \]

Hence:
\[ \lim_{\varepsilon \to 0} (x_\varepsilon)^{t_\varepsilon} = x(\pi - \theta)/\theta . \]
If we write \( x = C \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} C^{-1} \), we compute that:

\[ x'(\Pi-\theta)/\theta = C \begin{bmatrix} e^{-i(\Pi-\theta)} & 0 \\ 0 & e^{i(\Pi-\theta)} \end{bmatrix} C^{-1} \]

so that:

\[ x'(\Pi-\theta)/\theta = -x^{-1} \quad \square \]

Having established the claim, we conclude that:

\[ \lim_{\epsilon \to 0} f_\epsilon(\rho',x) = (\rho',-x^{-1} \cdot \rho') \]

\[ = (\rho',x^{-1} \cdot \rho') \]

\[ = f_-(\rho',x) \]

From steps (1), (2) and (3), we conclude that:

\[ \lim_{\epsilon \to 0} f_\epsilon = f_- \quad \square \]

Let:

\[ f : R'_+ \times S^3 \longrightarrow R(K) \]

\[ (\rho',u) \longmapsto (\rho',u \cdot \rho') \]

Get:

\[ \delta = \tilde{i}_* (\tilde{f}(R'_- \times S^3)) \in H_{sg}(\tilde{R}(K)) \]

Consider the natural map:

\[ i_* : H_{sg}(\tilde{R}(K)) \longrightarrow H_{sg}(\tilde{R}) \]

**Corollary 4.3:** The difference cycle \( \delta \) is given as:

\[ \delta = i_* (\tilde{f}_- (R'_- \times S^3)) \]

**Proof:** From Lemma 4.2, there is a continuous map:

\[ \tilde{F} : (R'_- \times S^3) \times [0,\alpha) \longrightarrow \tilde{R} \]

with:

\[ \tilde{F}_\epsilon = \tilde{F}_\epsilon \quad 0 < \epsilon < \alpha \quad \tilde{F}_0 = \tilde{F} \]

Since:

\[ \delta_\epsilon = -\tilde{F}_\epsilon (R'_- \times S^3) \quad 0 < \epsilon < \alpha \]

we conclude that \( \delta = i_* (\tilde{f}_- (R'_- \times S^3)) \).

But the inversion map on \( S^3 \) is orientation reversing:

\[ \text{inv} : S^3 \longrightarrow S^3 \]

\[ u \longmapsto u^{-1} \]

- This implies that the following map is orientation reversing:

\[ h : R'_- \times S^3 \longrightarrow R'_- \times S^3 \]

\[ (\rho',u) \longmapsto (\rho',u^{-1}) \]

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Moreover, the natural map is an $SO_3$ bundle map:

$$\tilde{\Pi} : R'_- \times S^3 \longrightarrow \tilde{R}'_- \times S^3.$$ 

Hence, since $h$ is $SO_3$ equivariant, $\tilde{h}$ is also orientation reversing.

- Clearly, $\tilde{\Pi} = \tilde{\Pi}_- \cdot \tilde{h}$.
- Hence, $\tilde{\Pi}(R'_- \times S^3) = \tilde{\Pi}_-(R'_- \times S^3)$ and $\delta = i_\delta(\delta_-)$.\u201d

\textit{d) The cycle $\delta_-$ in $\tilde{H}(\mathcal{R})$ as a pullback}

In this section, we give an alternative description of $\delta_-$ as a pullback of a fibre bundle with total space $\tilde{H}(\mathcal{R})$. We begin by considering the map $\tilde{\Pi}$ of the previous section:

$$\tilde{\Pi} : R'_- \times S^3 \longrightarrow \tilde{H}(\mathcal{R}) = R'_- \times R'_-$$

$$[(\rho', u_1)] \longmapsto [(\rho', u_1)']$$

Note: From hereon $\sim$ will denote equivalence class or orbit under the appropriate $SO_3$ action.

\textbf{Proposition 4.4:} $\tilde{\Pi}$ is a 2-fold cover of its image.

\textbf{Proof:} It suffices to show that $\tilde{\Pi}$ is 2-to-1.

- But $\tilde{\Pi}[(\rho'_1, u_1)] = \tilde{\Pi}[(\rho'_2, u_2)]$ iff $\rho'_2 = C \rho'_1 C$ and $u_2 \rho'_1 u_2 = C u_1 \rho'_1 u_1 C$ for some fixed $C \in S^3$.

- This latter condition, however, implies that:

$$\rho'_1 = (C^{-1} u_2^{-1} C u_1) \rho'_1 (u_1^{-1} C^{-1} u_2 C).$$

- Since $R'_-$ consists of irreducible representations and $\rho'_1 \in R'_-$, we conclude from Proposition 1.2.3 that:

$$C^{-1} u_2^{-1} C u_1 \in \sim 1.$$ 

- Hence, $\rho'_1 = C \rho'_1 C$ and $u_2 = C u_1 C^{-1}$.

- Thus, $[(\rho'_1, u_1)] = [(\rho'_1, C u_1 C^{-1})]$.  

- Now suppose that $[(\rho'_1, u_1)] = [(\rho'_1, u_1')]$. Then as above:

$$\rho'_1 = C \rho'_1 C^{-1}$$

- Once again, by Proposition 1.2.3, we conclude that:

$$C = \pm 1$$

- This is clearly impossible, and hence $\tilde{\Pi}$ is 2-to-1.\n
The previous proof indicates a natural factorization of $\tilde{\Pi}$ by a 2-fold cover and an inclusion. We define the following action of $SO_3$:

$$SO_3 \times (R'_- \times SO_3) \longrightarrow R'_- \times SO_3$$

$$(C, (\rho', A)) \longmapsto (C \rho', C A C^{-1})$$

We have the natural map:

$$\Pi : R'_- \times S^3 \longrightarrow R'_- \times SO_3$$

$$(\rho', A) \longmapsto (\rho', [A])$$

which induces a map of orbit spaces:

$$\tilde{\Pi} : R'_- \times S^3 \longrightarrow R'_- \times SO_3.$$
Finally, we can define a map:

\[ g : \hat{R}_+ \times S^2 \longrightarrow \hat{R}_-(K) \]

\[ (\rho', u) \longmapsto (\rho', u \cdot \rho') \]

which induces a map of orbit spaces:

\[ \hat{g} : \hat{R}_+ \times S^2 \longrightarrow \hat{R}_-(K) \].

**Corollary 4.5:** (1) \( \hat{f} = \hat{g} \cdot \hat{u} \)

(2) \( \hat{g} \) is an embedding

(3) \( \hat{u} \) is a 2-fold cover

(4) \( \delta_\_ = 2 \cdot \hat{g}(\hat{R}_- \times S^2) \)

**Proof:** This is an immediate consequence of Proposition 4.4. \( \square \)

Now we have an obvious projection map:

\[ p : \hat{R}_-(K) \longrightarrow \hat{R}_+ \times \hat{R}_- \]

\[ [(\rho', \rho'')] \longmapsto ([(\rho'), [\rho'']) \]

**Proposition 28:** \( p \) is a fibre bundle with fibre \( S^2 \).

**Proof:** It suffices to show that the fibre of \( p \) is \( S^2 \).

- Suppose \( p([(\rho', \rho'')] = [(\rho'), [\rho'']] \).

- Then \( \rho' = C \rho' C^{-1} \) and \( \rho' = D \rho'' D^{-1} \), \( C, D \in S^3 \).

- Hence, \( [(\rho', \rho'')] = [(\rho', E \rho'' E^{-1})], E = C^{-1} D \in S^2 \).

- In other words, \( p^{-1}([\rho'), [\rho'']) = \{[(\rho', E \rho'' E^{-1})] \mid E \in S^3 \}. \)

- But \( [\rho', E \rho'' E^{-1}] = [\rho', E \rho' E^{-1}] \) iff \( E = \epsilon E \).

- Hence, \( p^{-1}([\rho'), [\rho'']) \) is \( S^2 \). \( \square \)

Let \( \hat{\alpha} \) be the diagonal in \( \hat{R}_+ \times \hat{R}_- \).

**Proposition 4.7:** \( p^{-1}(\hat{\alpha}) = \hat{g}(\hat{R}_- \times S^2) \).

**Proof:** \( \hat{g}((\rho', u)) = p((\rho', u \cdot \rho')) = ([(\rho'), [\rho'']) \).

- Hence, \( \hat{g}(\hat{R}_- \times S^2) \subseteq p^{-1}(\hat{\alpha}) \).

- On the other hand, suppose \( p([(\rho', \rho'')] \in \hat{\alpha} \).

- Then \( [\rho'] = [\rho''] \) or \( \rho' = u \cdot \rho', u \in S^2 \). \( \square \)

Hence, we have a pullback diagram:

\[ \begin{array}{ccc}
\hat{R}_+ \times S^2 & \longrightarrow & \hat{R}_-(K) \\
p \downarrow \hat{g} & & \downarrow p \\
\hat{\alpha} & \longrightarrow & \hat{R}_+ \times \hat{R}_- \\
\text{inc} & & \\
\end{array} \]

**Corollary 4.8:** \( \delta_\_ = 2 \cdot p^{-1}(\hat{\alpha}) \)

**Proof:** This is an immediate consequence of the previous proposition. \( \square \)

**Note:** \( p^{-1}(\hat{\alpha}) \) has a natural orientation inherited from \( \hat{\alpha} \) via \( p \). We may think of this in two ways (at least).

1. **Duality**

   Suppose \( p : M \longrightarrow N \), where \( M \) and \( N \) are manifolds. Suppose \( \hat{\alpha} \) is an oriented submanifold of \( N \). Suppose \( p \) is transverse to \( \hat{\alpha} \). Then \( p^{-1}(\hat{\alpha}) \) is a submanifold of \( M \) with a natural orientation given by:
where $D$ is Poincaré duality.

(2) **Transversality**

Under the same assumptions as in (1), we may orient the tangent spaces of $p^*(\mathbb{A})$ compatibly by the rules:

$$T_x(p^*(\mathbb{A})) = T_x(F) \oplus T_x(\mathbb{A})$$

where $x = p(\mathbb{A})$ and $F_x = p^*(x)$. (Compare this with the discussion in section IV.1.)

These two descriptions are, of course, equivalent.

5. **Casson's invariant for fibred knots**

(a) **Fibered knot**

We recall that a knot $K$ in a 3-manifold $M$ is a fibered knot with monodromy $f$ if the knot space $M^3(K)$ fibres over $S^1$ with fibre $F'$ (where $F'$ is a Seifert surface for $K$) and monodromy $f$:

$$M^3(K) = M^3 \setminus \text{interior}(N(K))$$

$N(K) = \text{closed tubular nbd. of } K \text{ in } M^3$

$M^3(K) = (F' \times [-1,1]) / \sim \partial F' = K$

$\partial F'$ is homotopic to $K$ in $N(K)$

$$M^3(K) = \frac{F' \times J}{\langle (x,1) - (f(x),-1) \rangle} \quad J = [-1,1]$$

From this fibration of $M^3(K)$, we obtain a strongly preferred Heegaard decomposition for $K_D$, (as in Lemmas 1.2 and 1.3), as follows. Let:

$$W_1 = F' \times [0,1] \quad W_2 = F' \times [-1,0]$$

Let:

$$A_1 = \partial F' \times [0,1] \quad A_2 = \partial F' \times [-1,0]$$

Then:

$$\partial W_1 = (F' \times \{0\}) \cup A_1 \cup (F' \times \{1\})$$

$$\partial W_2 = (F' \times \{0\}) \cup A_2 \cup (F' \times \{-1\})$$

We consider the following homeomorphism:

$$q : \partial W_1 \quad \rightarrow \quad \partial W_2$$

$$(x,0) \quad \mapsto \quad (x,0) \quad x \in F'$$

$$(x,t) \quad \mapsto \quad (x,-t) \quad x \in \partial F' \quad 0 < t < 1$$

$$(x,1) \quad \mapsto \quad (f(x),-1)$$

Then we see that:

$$M^3 = \frac{W_1 \cup W_2}{\langle q(x) - x \mid x \in \partial W_1 \rangle}$$

If we identify $W_2$ with $W_1$ via the map:

$$r : W_2 \quad \rightarrow \quad W_1$$

$$(x,t) \quad \mapsto \quad (x,-t)$$

then we obtain an Heegaard model for $M^3$:

$$M^3 = (W,h) \quad W = F' \times I$$
where:

\[ J^W = F' \cup A \cup F^a \]

\[ F' = F' \times \{0\} \quad A = \partial F' \times 1 \quad F^a = F' \times \{1\} \]

\[ h : J^W \to J^W \]

\[ x \mapsto x \quad x \in F' \cup A \]

\[ x \mapsto f(x) \quad x \in F^a \]

\[ K = \partial F' \times \frac{1}{2} \]

**Note:** This decomposition satisfies Lemmas 1.2 and 1.3.

(b) Casson's invariant for fibered knots in homology 3-spheres

and Lefschetz numbers

Given a fibered knot \( K \) in an homology 3-sphere \( M \), let \((w,h)\) be an F-measured Heegaard decomposition for \( K_N \) as in the previous section. As in section 4(a), we may identify representation spaces as follows:

\[(R')^* = \mathbb{R}(\pi_1(F')) \oplus (S^2)^{2g_1} \quad g_1 = \text{genus}(F')\]

\[ R^* = \mathbb{R}(\pi_1(J^W \setminus \{0\})) = (R')^* \times (R')^* \]

\[ R = \mathbb{R}(\pi_1(J^W)) \subset R^* \]

\[ R = \{(\rho', \rho'') : \partial \rho' = \partial \rho''\} \]

\[ R = \{(\rho', \rho'') : \rho'(\lambda) = \rho''(\lambda)\} \]

\[ Q_1 = \{(\rho', \rho') : \rho' \in (R')^*\} \]

\[ Q_2 = \{(\rho', \tau(\rho')) : \rho' \in (R')^*\} \]

\[ S' = \text{reducible representations in } (R')^* \]

\[ S = \text{reducible in } R^* = S' \times S' \]

\[ R'_1 = \{\rho' : \rho' \in (R')^*, \partial \rho' = -I\} \]

\[ R_{-}(K) = R'_1 \times R'_1. \]

In addition, we have the following objects:

\[ Q_2^- = Q_2 \cap R_{-}(K) = \{(\rho', 1_{\mathbb{R}(\rho')}) : \rho' \in R'_1\} \]

\[ Q_2^- = \Gamma_{\mathbb{R}} = \text{graph of } f^\#: R'_1 \to R'_1 \]

\[ f^\#: \text{graph of } \hat{\rho} : R'_1 \to \hat{R}_1. \]

**Proposition 5.1:** \( Q_2^- \) is transverse to \( R_{-}(K) \) in \( R \).

**Proof:** First, we make a dimension count:

\[ \dim(R^*) = 6g_1 + 6g_1 = 12g_1 \]

\[ \dim(R) = 12g_1 - 3 \]

\[ \dim(Q_2^-) = 6g_1 \]

\[ \dim(R_{-}(K)) = (6g_1 - 2) + (6g_1 - 3) = 12g_1 - 6 \]

\[ \dim(Q_2^-) = 6g_1 - 3 \]

Hence, we see that:

\[ \dim(R) = \dim(Q_2^-) + \dim(R_{-}(K)) - \dim(Q_2^-) \]
Next, we compare tangent spaces at a point of intersection of \( Q_2 \) and \( R(K) \), \( x \):

\[
\mathbb{T}_x(Q_2) = \{(R, D_{x,t} f^x(x)) | \, x \in \mathbb{T}_x ((R')^x)\}
\]

\[
\mathbb{T}_x(R(K)) = \{(u, v) | \, u, v \in \mathbb{T}_{x_1} (R')\}
\]

\[
\mathbb{T}_x(Q_2^x) = \{(y, D_{x,t} f^x(y)) | \, y \in \mathbb{T}_{x_1} (R')\}
\]

Clearly, \( \mathbb{T}_x(Q_2^x) = \mathbb{T}_x(Q_2) \cap \mathbb{T}_x(R(K)) \).

The dimension count and intersection calculation establish transversality in \( R \). \( \Box \)

Note: We did not use the assumption that \( M \) is an homology 3-sphere in establishing this last proposition. This assumption is equivalent to the following condition on the monodromy:

\[
f_{x} - I : H_1(F', \mathbb{Z}) \longrightarrow H_1(F', \mathbb{Z}) \text{ is nonsingular.}
\]

As in the proof of Proposition III.1.1, we may interpret this in terms of \( D_{g_2} \).

But we made no assumptions about this derivative.

**Proposition 5.2:** \( \hat{\delta}_2 \) has an orientable normal bundle in \( \hat{G}_2 \).

**Proof:** By Proposition 5.1, \( \hat{\delta}_2 \) is a submanifold of \( \hat{G}_2 \), (as well as of \( \hat{R}_-(K) \)). All manifolds in question are orientable. \( \Box \)

We recall from sections 3 and 4 that:

\[
\lambda'(K) = \frac{1}{2} (s \cdot \hat{\delta}_2_{\hat{R}})
\]

\[
\epsilon_3 = 1_{\hat{x}}(\hat{\delta}_2_{\hat{R}}) \quad \delta_- = 2 \cdot \hat{g}((R' \times SO_3) = 2p^{-1}(\hat{\delta})
\]

\[
\hat{g} : (R' \times SO_3) \longrightarrow \mathbb{R}_- \times \mathbb{R}_-
\]

\[
[\rho', u] \longmapsto [\rho', u \cdot \rho']
\]

\[
p : R' \times R' \longrightarrow \hat{R}_- \times \hat{R}_-
\]

\[
[(\rho', \rho')] \longmapsto [\rho', \rho']
\]

\[
\tilde{\Lambda} = \text{diagonal of } \hat{R}_- \times \hat{R}_-
\]

Construct an isotopy of \( \tilde{\delta}_2 \) which carries \( \delta_- \) to a submanifold \( \tilde{\delta}_- \) with:

\[
\tilde{\delta}_- \text{ transverse to } \hat{\delta}_2_{-}.
\]

Extend this isotopy of \( \tilde{\delta} \) and let:

\[
\sigma = i_\hat{x}(\tilde{\delta}_-) \quad \hat{\sigma} = \text{isotopic to } \tilde{\sigma}.
\]

Then:

\[
\tilde{\sigma} \text{ is transverse to } \hat{\delta}_2 \quad \hat{\sigma} \text{ is transverse to } \hat{\delta}_2.
\]
Proof: It suffices to construct a section $\sigma$ of the restriction of $p$ to $\tilde{\Gamma}$:

$$p : \tilde{\Gamma} \to \Gamma$$

$$\{[\rho, f^*(\rho')]) \mapsto ([\rho_1', f^*(\rho')])$$

We need only to show that the obvious map is well defined:

$$\sigma : \Gamma \to \tilde{\Gamma}$$

$$\{[\rho], f^*(\rho')]) \mapsto ([\rho', f^*(\rho')])$$

But this is immediate from the observation:

$$\rho' = u' \rho'' \iff f^*(\rho') = u' f^*(\rho'')$$

We need the following simple lemma, which will allow us to compare:

$$<p^{-1}(A), \tilde{\Gamma}_{\tilde{R}_{-}}^* \tilde{R}_{-} \Gamma_{-}(K)$$

and

$$<A, \Gamma_{-} \times \tilde{R}_{-} \Gamma_{-}(K)$$

Lemma 6.4: Let $p : E \to B$ be a map of connected, oriented manifolds. Suppose that $A$ and $\Gamma$ are oriented submanifolds of $E$, that $p$ is transverse to $A$ and that $\Gamma$ lifts to an oriented submanifold of $E$, $\tilde{\Gamma}$. Then:

$$<p^{-1}(A), \tilde{\Gamma}_{-}^* \tilde{R}_{-} \Gamma_{-}(K) = <A, \Gamma_{-} \times \tilde{R}_{-} \Gamma_{-}(K)$$

Note: We have already used this in "projecting" to $\tilde{R}_{-}$ from $\tilde{R}$. The lemma follows readily from the same local transversality argument. However, we shall give an homological argument, via duality, for the sake of variety.
Proof: Note that, under the standard identifications of \( \pi_0(B) \) and \( \pi_0(B') \) with \( Z \), \( p_z \) is the identity:

\[
p_z = \text{id} : \pi_0(B', Z) \longrightarrow \pi_0(B, Z)
\]

\[
\langle p^{-1}(\Delta), \hat{\Gamma}\rangle_{\hat{R}} = p_\#(\pi^{-1}(\Delta)) \cap \hat{\Gamma} \in \pi_0(B, Z) = Z
\]

\[
p_\#(\pi^{-1}(\Delta)) \cap \hat{\Gamma} = p_\#(p_\#(\pi^{-1}(\Delta)) \cap \hat{\Gamma})
\]

\[
= \pi^{-1}(\Delta) \cap p_\#(\hat{\Gamma}).
\]

But \( \hat{\Gamma} \) is a lift of \( \Gamma \). That is, there is a section:

\[
\sigma : \Gamma \longrightarrow \hat{\Gamma} \quad p \circ \sigma = \text{id}.
\]

- Hence, \( p_\#(\hat{\Gamma}) = p_\# \sigma(\Gamma) = \Gamma \).
- Finally, \( p_\#(\pi^{-1}(\Delta)) \cap \hat{\Gamma} = \pi^{-1}(\Delta) \cap \Gamma. \)

Applying this last lemma, we conclude that:

\[
\langle \pi^{-1}(\Delta), \hat{\Gamma}\rangle_{\hat{R}(K)} = \langle \hat{\Delta}, \hat{\Gamma}\rangle_{\hat{R}_-^{-1} \hat{R}_-^+}.
\]

By definition, the right hand side is the Lefschetz number of \( \hat{\Delta} \):

\[
\langle \hat{\Delta}, \hat{\Gamma}\rangle_{\hat{R}_-^{-1} \hat{R}_-^+} = \mathcal{L}(\hat{\Delta} : \hat{R}_-^{-1} \hat{R}_-^+) = \mathcal{L}(\hat{\Gamma} : \hat{R}_-^{-1} \hat{R}_-^+).
\]

Corollary 6.6: If \( K \) is a fibered knot with monodromy \( f \) in an homology 3-sphere \( M \), then:

\[
\lambda'(K) = \lambda(\hat{\Gamma} : \hat{R}_-^{-1} \hat{R}_-^+) = \mathcal{L}(\hat{\Gamma} : \hat{R}_-^{-1} \hat{R}_-^+) = \mathcal{L}(\hat{\Gamma} : \hat{R}_-^{-1} \hat{R}_-^+) = \mathcal{L}(\hat{\Gamma} : \hat{R}_-^{-1} \hat{R}_-^+).
\]

(c) Fibered knots of genus one

We recall that a knot \( K \) in a 3-manifold \( M \) is a fibered knot of genus \( g_1 \) if \( K \) has a fibration such that the genus of the fiber is \( g_1 \).

Note: We do not assume that \( g_1 \) is minimal.

Suppose that \( g_1 = 1 \). Then:

\[
\pi_1(F') = \langle a, b \rangle = \text{free group of rank 2}.
\]

Consider the following representation:

\[
\rho_0 : \pi_1(F') \longrightarrow SU(2, \mathbb{C})
\]

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

Then:

\[
1_{\rho_0} = \rho_0(aba^{-1}b^{-1}) = -I
\]

\[
\rho_0 \in \mathbb{R}_{-1}^+
\]

Lemma 6.6: If \( g_1 = 1 \), then \( \hat{R}^{-1} = \{[\rho_0]\} \).

Note: By a dimension count, we know that \( \hat{R}^{-1} \) is a manifold of dimension 0. Moreover, we know it is connected, since it is a quotient of \( \mathbb{R}^+ \) and \( \mathbb{R}^+ \) is a retraction of the connected space \((\mathbb{R}')^4 \setminus S'):\n
\[
\dim((\mathbb{R}')^4) = 6
\]

\[
\dim(S') = 4
\]
Corollary 5.7: If $K$ is a fibered knot of genus one in an homology 3-sphere, then $|\lambda'(K)| = 1$.

**Proof**: This is an immediate consequence of Corollary 5.5 and Lemma 5.6. \(\Box\)

Corollary 5.8: Let $K$ be the trefoil or figure eight knot in $S^3$. Then $\lambda'(K) = \pm 1$.

**Note**: Since $\lambda$ and $\lambda'$ have only been defined up to a choice of sign $s$, we now choose $s$ such that $\lambda'(\text{trefoil knot}) = 1$.

We have not attempted to calculate $s$.

6. Dehn surgery on unlinks

(a) Dehn surgery on unlinks

Let $M^3$ be an homology sphere. Let $(K,L)$ be a link of two components in $M^3$. Then:

$$M^3(K,L) = M^3 \setminus \text{interior}(B(K \cup L)).$$

If $(\mu,\lambda)$ and $(\mu',\lambda')$ are standard meridian-longitude pairs for $K$ and $L$, respectively, then:

$$H_1(M^3(K,L)) = \mathbb{Z}_\mu \oplus \mathbb{Z}_{\mu'}.$$

Let $M^3(K,L; p/q, r/s)$ denote the manifold obtained from $M^3$ by performing $p/q$ surgery to $K$ and $r/s$ surgery to $L$ (as in section 1(a)). This is an homology 3-sphere precisely when $p = r = 1$ (assuming $p, r > 0$). As before, we denote these homology spheres as follows:
Now suppose that $K$ and $L$ have zero linking number. That is to say:

$$\lambda = \lambda' = 0 \quad \text{in} \quad H_1(M^3(K,L)).$$

Now $K$ is naturally contained in $L_n$. This assumption implies that $(\mu, \lambda)$ is a standard meridian longitude pair for $K$ in $L_n$. Hence, it is clear that:

$$(K_n, L_n) = L_n(\frac{\mu}{n}, \frac{\lambda}{n}) = K_n(\frac{\mu}{n}, \frac{\lambda}{n}).$$

**Note:** In general, $(K_m, L_n)$ is $L_n(K, p/q)$ for some surgery coefficient $p/q$ which depends on $m$, $n$ and the linking number of $K$ and $L$.

(b) **Preferred Heegaard decompositions for surgeries on boundary links**

Suppose that $(K,L)$ is a boundary link. That is, $K$ and $L$ bound disjoint Seifert surfaces. (In particular, $K$ and $L$ have zero linking number.) In a manner similar to the construction of section 1(b), we may construct a particularly convenient Heegaard decomposition compatible with the various surgeries on $(K,L)$.

**Proposition 6.1:** There exists an Heegaard decomposition for $M^3(W_1, W_2)$ such that:

$$W_1' = F' \times I \quad L' = JF' \times (\frac{1}{2})$$

and $K$ is a separating curve in $F'$.

**Proof:** Choose a pair of disjoint Seifert surfaces for $K$ and $L$, $F_0$ and $F_1$. Thicken $F_0$ to $F_0 \times I$ with $F_0 \times 0$ disjoint from $F_1$. Let $F_2$ be the boundary of $F_0 \times I$ and form the connected sum of $F_2$ and $F_1$ by tubing $F_1$ and $F_2$ together. By choosing the tubing away from $K$ and $L$, we obtain a surface, $F'$, with the following properties:

$$L = \partial F' \quad K \text{ is separating in } F'.$$

The rest of the proof continues along the lines of the proofs of Lemmas 1.1 and 1.2. In order to preserve the condition that $K$ is separating, we only need to ensure that all $1$-handles are attached in the complement of the component of $F' \setminus K$ bounded by $K$. This is easily accomplished by sliding the attaching discs off this surface.

As in section 1(b), we identify this decomposition with an Heegaard model $(W_1, W_2)$. Let $v_K$ and $v_L$ denote the Dehn twists about $K$ and $L$ respectively. We obtain the analog of Proposition 1.3.
Proposition 5.2: \((W, \text{h}, \pi_{n+1}^M, \bar{\pi}_L)\) is an Heegaard model for \((K, L, n)\).

7. Casson's invariant for unlinks

(a) Casson's invariant for unlinks \(\lambda^*(K, L)\)

Let \((K, L)\) be an unlink of two components in an homology 3-sphere \(M^3\). That is, assume that \(K\) and \(L\) have zero linking number. As previously observed, \(K\) is naturally contained in \(L_n\), \(L\) is naturally contained in \(K_m\). We denote these various knots as follows:

\((K, L_n) = K\) in \(L_n\), \((K_m, L) = L\) in \(K_m\).

Similarly, we use \(\lambda^*(K, L_n)\) and \(\lambda^*(K_m, L)\) to denote Casson's invariant for the knots \((K, L_n)\) and \((K_m, L)\) respectively.

Casson's invariant for the unlink \((K, L)\) in \(M^3\) is defined by the rule:

\[
\lambda^*(K, L) = \lambda(K_{n+1}, L_{n+1}) - \lambda(K_{n+1}, L_{n+1})
- \lambda(K_{n+1}, L_n) + \lambda(K_{n+1}, L_n)
\]

By grouping the terms in two distinct ways, we see:

\[
\lambda^*(K, L) = \lambda^*(K, L_{n+1}) - \lambda^*(K, L_n)
- \lambda^*(K, L_{n+1}) + \lambda^*(K, L_n)
\]

This shows that \(\lambda^*(K, L)\) is independent of \(n\) and \(m\) respectively. Hence, \(\lambda^*(K, L)\) is well defined.

(b) Triviality for boundary links

Let \((K, L)\) be a boundary link in an homology sphere \(M^3\). Choose an

Heegaard decomposition for \(M^3\) as in Proposition 5.1 with corresponding Heegaard models as in Proposition 6.2.

Following the argument of section 2(b), we conclude that:

\[
\lambda(K_{n+1}, L_n) = \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle - \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle + \lambda(K_1, L_1)
- \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle - \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle + \lambda(K_1, L_1)
\]

Let \(h_K = \tau^{-1}\) and \(h_L = \tau^{-1}\). Then:

\[
\lambda^*(K, L) = \lambda(K_{n+1}, L_n) - \lambda(K_{n+1}, L_n) - \lambda(K_1, L_1) + \lambda(K_1, L_1)
- \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle - \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle - \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle - \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle + \lambda(K_1, L_1)
\]

Recall that we have a difference cycle:

\[
\delta_L = \tau_L(\delta_1^L) - \delta_1^L, \quad \tau_L(\delta_1^L) = \delta_L
\]

Then:

\[
\delta_L = \delta_L(\delta_1^L) = \delta_1^L - \delta_L(\delta_1^L)
\]

Hence, we conclude that:

\[
\lambda^*(K, L) = \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle - \frac{1}{2} \langle \delta_1^L, \delta_1^L \rangle + \lambda(K_1, L_1)
\]

Proposition 7.1: \(h_K(\delta_L) = \delta_L\) (rationally).

Proof: Recall that \(\delta_L\) is carried by \(\hat{R}_n(L_1)\) (section 2(c)). Here, we have:

\[
R_{n+1}(L_1) = R_{n+1} \times R_{n+1}'
\]

Let \(F'\) be the component of \(F\setminus L\) which contains \(K\) and denote the restriction of \(h_k\) to \(F'\) by \(h_k\). Then the action of \(h_k\) on \(R_{n+1}(L_1)\) is given by the rule:
\[ h^\#_R : R(-L) \rightarrow R(-L) \]
\[ (\rho', \rho'') \mapsto \langle h^\#_R(\rho'), \rho'' \rangle \]

In addition, we have the projection:
\[ p : \hat{R}_-(L) \rightarrow \hat{R}_- \times \hat{R}_- \]

From Corollary 4.8, we know that:
\[ \delta_L = 2p^{-1}(\hat{\lambda}) \]

Here \( \hat{\lambda} \) is the diagonal of \( \hat{R}_- \times \hat{R}_- \). In terms of Poincaré duality, this equality may be expressed as follows:
\[ D^{-1}(\delta_L) = 2 \cdot p^*(D^{-1}(\hat{\lambda})) \]

From the results of the next chapter (Theorem VI.2.4) and the fact that \( h^\#_R \) acts trivially on the homology of \( F' \), we conclude that:
\[ (\tilde{h}_R)_\# = \text{id} : H^R_-(\hat{R}_-; R) \rightarrow H^R_-(\hat{R}_-; R) \]

Hence, by the Kunneth formula and the fact that \( h^\#_R \) acts trivially on the second factor:
\[ (\tilde{h}_R)_\# = \text{id} : H^R_-(\hat{R}_- \times \hat{R}_-) \rightarrow H^R_-(\hat{R}_- \times \hat{R}_-) \]

There \( \tilde{h}_R \) is the obvious map induced from \( h_R \). Since the rational homology and cohomology are dual:
\[ (\tilde{h}^*_R)_\# = \text{id} : H^R_-(\hat{R}_- \times \hat{R}_-) \rightarrow H^R_-(\hat{R}_- \times \hat{R}_-) \]

Since the following diagram commutes:

\[ H^k(\hat{R}_- \times \hat{R}_-) \xrightarrow{P^*} H^k(\hat{R}_- \times \hat{R}_-) \]
\[ \tilde{h}^*_R(\hat{R}_- \times \hat{R}_-) \xrightarrow{P^*} H^k(\hat{R}_- \times \hat{R}_-) \]

we conclude from the previous remarks that:
\[ (\tilde{h}^*_R(\hat{R}_- \times \hat{R}_-))^* = D^{-1}(\delta_L) \]
\[ p^* \]
\[ H^k(\hat{R}_- \times \hat{R}_-) \xrightarrow{P^*} H^k(\hat{R}_- \times \hat{R}_-) \]

\[ (\tilde{h}^*_R(\hat{R}_- \times \hat{R}_-))^* = D^{-1}(\delta_L) \]

\[ \tilde{h}^*_R(\hat{R}_- \times \hat{R}_-) \]

\[ \delta_L \]

Corollary 7.2: If \( (K,L) \) is a boundary link in an homology 3-sphere \( M^3 \), then \( \lambda^*(K,L) = 0 \).

Proof: This follows immediately from the previous identity:
\[ \lambda^*(K,L) = \frac{1}{2} \langle \delta_L, \hat{\lambda}_2 \rangle - \langle \tilde{h}_R(\delta_L), \hat{\lambda}_2 \rangle \]

and Proposition 7.1. \( \square \)

(c) Casson's invariant for unlinks. \( \lambda^*(K,C,C') \)

Let \( (K,C,C') \) be an unlink of three components in an homology 3-sphere \( M^3 \). That is, assume that every pair of components has zero linking number.

As in previous sections, we adopt notation:
(K_{n}, C_{k}, C'_{m}) = M_{p}(K, C, C'; \frac{1}{n}, \frac{1}{k}, \frac{1}{m})
(K_{n}, C_{k}, C') = C' \text{ in } (K_{n}, C_{k})
(K_{n}, C', C') = (C, C') \text{ in } K_{n}

and the various modified forms.

Casson's invariant is then defined by the rule:

\[ \lambda^{n}(K, C, C') = \lambda(K_{n+1}, C_{k+1}, C'_{m+1}) - \lambda(K_{n+1}, C_{k}, C'_{m}) 
- \lambda(K_{n+1}, C_{k+1}, C'_{m}) + \lambda(K_{n+1}, C_{k}, C'_{m}) 
- \lambda(K_{n}, C_{k+1}, C'_{m+1}) + \lambda(K_{n}, C_{k}, C'_{m}) 
+ \lambda(K_{n}, C_{k+1}, C'_{m}) - \lambda(K_{n}, C_{k}, C'_{m}) \]

\( \gamma \) grouping the terms in various ways, we see:

\[ \lambda^{n}(K, C, C') = \lambda^{n}(K_{n+1}, C_{k}, C'_{m}) - \lambda^{n}(K_{n}, C, C') 
= \lambda^{n}(K, C_{k+1}, C') - \lambda^{n}(K, C_{k}, C') 
= \lambda^{n}(K, C_{k}, C'_{m+1}) - \lambda^{n}(K, C_{k}, C'_{m}) . \]

The fact that all these terms are defined follows from the assumption at each pair of components is unlinked. This is an easy exercise which we leave to the reader.

These various formulations show that \( \lambda^{n}(K, C, C') \) is well defined independently of \( n, k \) and \( m \).

8. Casson's invariant and the Alexander polynomial

(a) The Alexander polynomial \( A_{K}(t) \)

Let \( K \) be a knot (or link) in an homology 3 sphere \( M^{3} \). Let \( F \) be a Seifert surface in \( M^{3} \) for \( K \). Let \( V \) be a matrix for the linking form:

\[ < , > : H_{1}(F, \mathbb{Z}) \times H_{1}(F, \mathbb{Z}) \rightarrow \mathbb{Z} \]

\[ (x, y) \rightarrow \text{link}(x, y^{*}) \]

where \( y^{*} \) is the "plus push off" of \( y \) ((RoI)). Then we can define the Alexander polynomial of \( K \) by the rule:

\[ A_{p}(t) = \det(V - tv^{*}) . \]
If we change the Seifert surface, the polynomial changes by at most a unit multiple:

\[ \Delta_{\mathcal{F}}(t) = t^n \Delta_{\mathcal{F}'}(t) \quad n \in \mathbb{Z}. \]

We shall denote any such polynomial as \( \Delta_k \). We recall that \( \Delta_k \) is symmetric up to a unit:

\[ \Delta_k(t^{-1}) = t^{-r} \Delta_k(t) \]

where \( r \) is the rank of \( H_1(F, \mathbb{Z}) \). In particular, if \( K \) is a knot, \( r = 2g \), where \( g \) is the genus of \( F \). Hence, we obtain a symmetrized Alexander polynomial by symmetrizing:

\[ \Delta_k(t) = t^{r/2} \Delta_k(t) \]

\[ \Delta_k(t^{-1}) = \Delta_k(t) \]

\[ \Delta_k(t) = a_0 + a_1(t + t^{-1}) + \cdots + a_m(t^m + t^{-m}) . \]

We recall that \( \nu - \nu^T \) is the intersection form:

\[ \nu - \nu^T = (\ , )_F : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \rightarrow \mathbb{Z} . \]

Hence, we have the following proposition:

**Proposition 8.1:** Let \( K \) be a knot or link. Then:

\[ \Delta_k(1) = 1 \quad \text{if } K \text{ is a knot} \]

\[ \Delta_k(1) = 0 \quad \text{if } K \text{ is not a knot} . \]

---

**Proof:** For a connected surface of one boundary component, the intersection form is the standard symplectic form which has determinant one. On the other hand, a connected surface of more than one boundary component has a degenerate intersection form, since any boundary component represents a nontrivial element of the kernel. \( \square \)

**Note:** This and subsequent results are well-known for knots and links in \( S^3 \). In particular, they follow from results in Knifman [K] for the Conway polynomial.

**Corollary 8.2:** Let \( K \) be a knot with Seifert surfaces \( F \) and \( F' \). Then:

\[ \Delta_{\mathcal{F}}(t) = t^n \Delta_{\mathcal{F}'}(t) \quad n \in \mathbb{Z} . \]

**Note:** According to the classical definition of the Alexander polynomial as an elementary divisor, it is only well defined up to a unit. However, for any link in \( S^3 \), any two Seifert matrices are related by the moves described in [Bi], pg. 236. Hence, the above corollary also holds for links in \( S^3 \).

(b) **The second derivative of \( \Delta_k(t) \)**

We have the following identities for links of two components in \( S^3 \):

**Proposition 8.3:** If \( K \) is a link of two components in \( S^3 \) and \( I(K) \) is the linking number of these components, then:

\[ \Delta_k'(1) = I(K) . \]

**Note:** The conclusion is independent of the choice of Seifert surface. As we have previously observed, since \( K \) is a link in \( S^3 \), if \( F' \) is another Seifert surface for \( K \), then:
\[ \Delta_p(t) = e^{\Delta_g(t)} \quad n \in \mathbb{Z} \]

Hence, we compute that:

\[ \Delta_p'(1) = n \cdot \lambda_g(1) + \lambda_g'(1) = \lambda_g'(1) \]

**Proof of Proposition 8.3**

The proof is by induction on the geometric linking number of the components of $K$ (i.e., the minimal number of elementary crossings of the components of $K$ required to homotope $K$ to a splittable link).

Hence, we first consider the situation where $K$ is a splittable link $K_1, K_2$. Let $S^2$ be a separating 2-spheres and $F_1$ and $F_2$ be Seifert surfaces for $K_1$ and $K_2$ respectively with $F_1$ and $F_2$ contained in components of $S^3 \backslash S^2$.

![Figure 31](image1)

Tubing $F_1$ and $F_2$ together along an arc in $S^3 \backslash (F_1 \cup F_2)$, we obtain Seifert surface for $K_1, K_2$.

The corresponding Seifert matrix has the form:

\[
 V = F_1 \begin{bmatrix} v_1 & 0 & 0 \\ v_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Hence, $\lambda_g(t) = 0$. On the other hand, $\tau(K) = 0$. This establishes the result for split links.

Hence, we may assume, by standard arguments, that there is a projection of $K$ with at least one elementary linking of the components of $K$, such that an elementary crossing at this elementary linking reduces the geometric linking of $K$.
Using the standard Seifert surface for this projection, F', we can choose a basis for $H_1(F',\mathbb{Z})$, $(a_1, b_1, \ldots, a_n)$, containing curves $a$ and $b$ as depicted in Figure 34.

Note: If the associated surface for $K'$, $F'$ is not connected, then we can tube the two components together. By adding this tube as a handle to $F$, we insure that the curve $b$ exists.

Then the corresponding Seifert matrix for $K$ has the form:

$$
\begin{bmatrix}
    a^+ & b^+ & (F')^+ \\
    a & -1 & -1 & 0 \\
    b & 0 & -1 & * \\
    F' & 0 & * & V'
\end{bmatrix}
$$

where $V'$ is the Seifert matrix for $F'$. Then we compute that:

$$
V - tv^T = \begin{bmatrix}
    (t - 1) & -1 & 0 \\
    t & (t - 1) & * \\
    0 & * & V' - t(V')^T
\end{bmatrix}
$$

We observe that the matrix obtained by deleting the first row and column is an Alexander matrix for the knot $L$ obtained by slicing $F$ transverse to $a$ (i.e. by deleting the generator $a$):

Figure 35
Hence, we have the following identity:
\[ \tilde{\Delta}_K(t) = (t - 1)\tilde{\Delta}_L(t) + t\tilde{\Delta}_K(t). \]

Computing derivatives, we conclude that:
\[ \tilde{\Delta}_K'(1) = \tilde{\Delta}_L(1) + \tilde{\Delta}_K'(1) + \tilde{\Delta}_K'(1). \]

From the previous section, we know that:
\[ \tilde{\Delta}_L(1) = 1 \quad \tilde{\Delta}_K'(1) = 0. \]

By induction, we conclude that:
\[ \tilde{\Delta}_K'(1) = \#(K'). \]

On the other hand, we readily see that:
\[ \#(K) = \#(K') + 1. \]

Thus, we have the desired result:
\[ \tilde{\Delta}_K'(1) = \#(K). \]

Now suppose that $K_-, K_+$ and $K_o$ are links in $S^3$ which have projections which differ at a single crossing of $K_-$ as depicted below.

\[ K_- \quad K_+ \quad K_o \]

Figure 36

Lemma 8.4: Let $K_-$ be a knot in $S^3$. Let $K_+$ and $K_o$ be as above. Then the link $K_o$ is a link of two components, and:
\[ \frac{1}{2} \tilde{\Delta}_{K_+}'(1) - \frac{1}{2} \tilde{\Delta}_{K_-}'(1) = \#(K_o). \]

Proof: $K_o$ has two components for homological reasons. Now choose the Seifert surfaces as follows:

\[ F_- \quad F_+ \quad F_o \]

\[ K_- \quad K_+ \quad K_o \]

Figure 37

So we can arrange the corresponding Seifert matrices $V_+, V_-, V_o$ as follows:

\[ V_+ = \begin{bmatrix} a_{11} & x \\ z & V_o \end{bmatrix} \quad V_- = \begin{bmatrix} a_{11} + 1 & x \\ z & V_o \end{bmatrix} \]

\[ \tilde{\Delta}_+(t) = \det(V_+ - tv_+^T), \quad \tilde{\Delta}_-(t) = \det(V_- - tv_-^T), \quad \tilde{\Delta}_o(t) = \det(V_o - tv_o^T), \quad \text{hence:} \]
\[ \tilde{\Delta}_o(t) - \tilde{\Delta}_o(t) = (t - 1)\tilde{\Delta}_o(t). \]

Observe that $F_-$ is homeomorphic to $F_+$. Hence, genus($F_-) = \text{genus}(F_+) = g$.

To obtain the symmetrized Alexander polynomials, we have to multiply by $t^{-g}$. Hence:
\[ \Delta_K(t) - \Delta_\zeta(t) = (t-1) t^{-2} \Delta_\zeta(t) . \]

Let \( \psi(t) = (t-1) t^{-2} \Delta_\zeta(t) \). Computing derivatives:

\[ \psi'(1)/2 = -\hat{\Delta}_\zeta(1) + \hat{\Delta}_\zeta'(1) . \]

Hence, by Propositions 8.1 and 8.3:

\[ \frac{1}{2} \Delta_{K'}(1) - \frac{1}{2} \Delta_{K''}(1) = \#(K) . \]

(c) Casson's invariant and the Alexander polynomial

We are now ready to prove the main result relating Casson's invariant to the classical Alexander polynomial. We begin with a simple lemma.

**Lemma 8.5:** Let \( K \) be a knot in an homology 3-sphere \( M^3 \). Then there exists a knot \( L \) in \( S^3 \) such that \( \lambda'(K) = \lambda'(L) \) and \( \Delta_K(t) = \Delta_L(t) \).

**Proof:** It is well known that \( S^3 \) may be obtained by \( \pm 1 \) surgeries on the components of a link in \( M^2 \):

\[ S^3 = (C_1, \ldots, C_n) \]

\[ C = (C_1, \ldots, C_n) \text{ is a link in } M^2 . \]

The proof is by induction on \( n \). It is obvious for \( n = 0 \).

Now choose Seifert surfaces for \( K \) and \( \zeta \). \( \zeta_1, \zeta' \) and \( \zeta'' \). By general position, we may assume that there exist one dimensional spines, \( \Sigma \) and \( \Sigma' \), for \( \zeta_1 \) and \( \zeta'' \) which are disjoint. We may isotope \( \zeta_1 \) into a small regular neighborhood of \( \Sigma \). By a separate isotopy, we may isotope \( \zeta'' \) into a small regular neighborhood of \( \Sigma' \). Hence, we may assume that \( \zeta_1 \) and \( \zeta'' \) are disjoint. (Of course, this may require crossings of \( K \) and \( \zeta \), but not self crossings of \( \zeta \).) In other words, we may assume that \( (K;\zeta\zeta) \) is a boundary link in \( M^3 \). Hence, \( \lambda'(K;\zeta\zeta) = 0 \) by Corollary 7.2. That is:

\[ \lambda'(K) = \lambda'(K;\zeta\zeta) . \]

On the other hand, since \( K \) and \( \zeta \) bound disjoint Seifert surfaces in \( M^3 \setminus (K;\zeta) \), we may also conclude that:

\[ \Delta_K(t) = \Delta_{(K;\zeta)}(t) \]

where \( (K;\zeta\zeta) \) is the knot \( K \) considered as a knot in the surgered manifold \( M^3 \setminus (\zeta;\zeta) \). If we denote this manifold by \( N^3 \):

\[ S^3 = (C_1, \ldots, C_n) \]

\[ D = (C, \ldots, C_n) \text{ is the link } (C_1, \ldots, C_n) \]

considered as a link in \( N^3 \).

Hence, by induction, there is a knot \( L \) in \( S^3 \) such that:

\[ \lambda'(K;\zeta\zeta) = \lambda'(L) \]

\[ \Delta_{(K;\zeta\zeta)}(t) = \Delta_L(t) . \]

This establishes the desired result. \( \square \)

Next we prove an analog of Lemma 8.4.

**Lemma 8.6:** Let \( K_+ \) be a knot in \( S^3 \). Let \( K_+ \) and \( K_\zeta \) be as above. Then \( K_\zeta \) is a link of two components and:
\( \lambda'(K_1) - \lambda'(K_-) = \#(K_0) \).

**Proof:** Suppose that \( K \) is a knot in \( S^3 \). Now suppose that \((C,C')\) is a link in \( S^3 \backslash K \) such that \( C \) and \( C' \) bound disjoint discs, \( D \) and \( D' \), such meeting \( K \) geometrically twice and algebraically zero times. In addition, assume that \( K \cap D \) and \( K \cap D' \) do not separate each other in \( K \):

![Figure 38](image)

In particular, \((C,C')\) is a boundary link in \( K_n \), for \( C' \) and \( C \) bound obvious disjoint punctured tori in \( S^3 \backslash K \):

![Figure 39](image)

Hence, \( \lambda'(K_n,C,C') = 0 \) for all \( n \). From this, we conclude that:

\[ \lambda'(K_n,C,C') = 0. \]

This in turn implies that:

\[ \lambda'(K_n,C,C') = \lambda'(K_n,C') \]

This means that changing \( K \) with a twist across \( D \) does not change the value of \( \lambda'(K_n,C') \). But, by changing \( K \) across such discs \( D \), we can change \( K \) to the following knot \( K^0 \):

![Figure 40](image)

Let \( C' \) denote the curve depicted below:

![Figure 41](image)
We have the following computation:

\[ \lambda'(K,C') = \lambda'(K^m,C') \]

\[ = \lambda'(K^m;C') - \lambda'(K^m;C_0') \]

\[ = \lambda'(K^m;C') \]

\[ = \sum_{j=1}^{m} (\lambda'(K^j;C'_j) - \lambda'(K^{j-1};C'_j)) \]

\[ = \sum_{j=1}^{m} (\lambda'(K^j;C'_j,C''_j)) \]

\[ = \sum_{j=1}^{m} (\lambda'(K^j;C'_j,C''_j)) \]

\[ = m(\lambda'(K',C''_1) - \lambda'(K^0;C'_0,C''_0)) \]

\[ = m(\lambda'(K';C'_0)) \]

\[ = m(\lambda'(trefoil\ knot)) = +m \]

\[ \lambda'(K,C'_1) - \lambda'(K,C'_0) = +m \]

\[ \lambda'(K;C'_0) - \lambda'(K) = +m \]

We may assume that \( K \) is \( K_+ \) and \( K_- \) is the knot obtained from \( K \) by applying a left twist across \( D' \). That is \( K_- = (K;C'_1) \). Let \( K_0 \) denote the two component link obtained from \( K \) by cutting across \( D' \) and connecting "the other way".

![Figure 43](image)

The move of twisting across the disc \( D \) only affects one component of \( K_0 \). Hence, it clearly leaves \( \#(K_0) \) invariant. For the standard form \( K^m \), however, it is evident that \( \#(K^m) = -m \):

![Figure 44](image)

Hence, from the previous comments:
It follows that:

\[ \lambda'(K_+^-) - \lambda'(K_-^-) = \varepsilon \cdot \sharp(K_0^-) \]
\[ \varepsilon = \lambda'(\text{trefoil knot}) . \]

As previously observed, we may "normalize" \( \lambda' \) so that:

\[ \lambda'(\text{trefoil knot}) = 1 . \]

For this normalization, we have the identity:

\[ \lambda'(K_+) - \lambda'(K_-) = \sharp(K_0) . \]

We now turn to the main theorem.

**Theorem 3.7:** If \( K \) is a knot in an homology 3 sphere \( M^3 \), then:

\[ \lambda'(K) = \frac{1}{2} \lambda^*(1) = \sum_{n=1}^{\infty} n \sum a_n \]

where \( \Delta_g(t) = a_0 + a_1(t + t^{-1}) + \cdots + a_m(t^m + t^{-m}) . \)

**Proof:** By Lemma 3.5, we assume that \( K \) is a knot in \( S^3 \). The proof is by induction on the number of crossings in a projection of \( K \) which must be changed in order to convert \( K \) to an unknot. Since the theorem is true for the unknot, we may consider a projection of \( K \) and a single crossing which we need to change. Let \( K_- \) denote \( K \), \( K_+ \), the knot obtained by changing his crossing and \( K_0 \) the usual associated link of two components. Then:

\[ \lambda'(K_+) - \lambda'(K_-) = \sharp(K_0) \]

\[ \frac{1}{2} \lambda^*_K(1) - \frac{1}{2} \lambda^*_K(1) = \sharp(K_0) . \]

On the other hand, by induction:

\[ \lambda'(K_+) = \frac{1}{2} \lambda^*_K(1) . \]

The result follows immediately. \( \square \)

With this result, we are able to establish property (5) for \( \lambda(M) \). Let \( \psi(K) \) be the arf invariant of \( K \).

**Property (5):** \( \lambda(M^3) \equiv \mu(M^3) \pmod{2} \)

**Proof:** From Theorem 7 of [R], (after symmetrizing), and Theorem 3.7 above, we conclude that:

\[ \psi(K) = \lambda'(K) \pmod{2} \]

for any knot in \( S^3 \). The argument of Lemma 3.5 permits us to deduce the same fact for any knot in any homology sphere. (The arf invariant also comes from a Seifert matrix.)

On the other hand, Theorem 2 of [G] implies that:

\[ \psi(K) \equiv \mu(K_1) - \mu(K_0) \pmod{2} . \]

This yields the identity:

\[ \lambda(K_1) - \lambda(K_0) \equiv \mu(K_1) - \mu(K_0) \pmod{2} . \]

Since we can write \( M^3 \) as a sequence of \( \pm 1 \) surgeries on a link in \( S^3 \):

\[ M^3 = S^3(C_1, \ldots, C_k) \]

and we know that:

\[ \lambda(S^3) = \mu(S^3) = 0 \pmod{2} \]

the property follows by induction on \( n \) using the identity above. \( \square \)
CHAPTER VI: THE TOPOLOGY OF THE SPACE OF REPRESENTATIONS

1. The topology of $R^g$

(a) Identification of spaces

As in previous sections, we identify $R^g$ with $(S^3)^g$. Since we shall be
inducting on $g$ at various points, we denote the boundary map as follows:

$\bar{a}^g : R^g \rightarrow S^3$

$(A_1, \ldots, A_g, B_1, \ldots, B_g) \mapsto \prod_{i=1}^g [A_i, B_i]$.

Likewise, we use the following notation:

$R_+^g = R^g \cap \partial^1(I)$

$N_+^g = \partial^1(D_+^3)$

$R_-^g = R^g \cap \partial^{-1}(I)$

$N_-^g = \partial^{-1}(D_-^3)$.

Clearly:

$R^g = R_+^g \cup N_+^g \cup R_-^g \cup N_-^g$.

The singular set of $\bar{a}^g$ is contained in $R_+$. Hence, $\bar{a}^g$ is regular
on $R^g \setminus S$. In particular, we have a fibre bundle with fibre $R_+$:

$\bar{a}^g : N_+^g \rightarrow D_+^3$.

Figure 45

Hence, there is a trivialization:

$w_g : (R^g_x \times U^2, R^g_x \times S^3) \rightarrow (N_x^g, N^g_x)$

with the following properties:

(1) $w_g(\bar{a}^g) = P_z$ (projection onto second factor)

(2) $w_g|_{R^g_x \cap (-1)}$ is the usual inclusion $R^g_x \hookrightarrow N_x^g$.

(b) The topology of $R^g$

Lemma 1.1: $R^g$ is connected for $g = 1$

simply connected for $g \geq 2$.

Proof: The restriction of the boundary map:

$\bar{a}^g : (S^3)^g \setminus R^g \rightarrow S^3 \setminus [I]$.

is a locally trivial fibration with fiber $R_+^g$. It suffices to show that
$(S^3)^g \setminus R^g$ is connected for $g = 1$ and simply connected for $g > 2$.

Consider the decomposition of $R^g_+$:

$R^g_+ = R_+ \cup R_1$, $R_+ = R \setminus S$, $R_1 = S$.

Let $\Lambda$ be $S^1$ in $S^3$ (i.e., $\Lambda$ is the set of diagonal matrices). Consider
the map:

$\alpha : \Lambda^g \times S^3 \rightarrow (S^3)^g$

$((A_1, \ldots, A_g, B_1, \ldots, B_g), T) \mapsto (TA_1^{-1}, TB_1^{-1}, \ldots, TA_g^{-1}, TB_g^{-1})$.

Then:

$\text{im}(\alpha) = R^g_1$.
Since diagonal matrices commute, a factors to:

\[ \alpha : (S^3/\Lambda) \to (S^3)^2 \]

Hence, \( \dim(R_+) = 2g + 2 \), which implies \( \text{codim}(R_+) = 4g - 2 \). Since \( \text{codim}(R_+) = 3 \), the result follows by general position.

We now want to relate the topology of \( R^{g+1}_- \) to the topology of \( R^g_+ \).

Write:

\[ (S^3)^2 = (S^3)^2 \times (S^3)^2 \]

Let \( P_+, P_- \) be the projections onto \( (S^3)^2 \) and \( (S^3)^2 \). So we can write:

\[ \partial_{g+1} = \partial_+ \circ P_+ \circ \partial_- \circ P_- \]

\[ R^{g+1}_- = \{(y_1, y_2) \in (S^3)^2 | \partial_{g+1}(y_2) = -\partial_+(y_1)^{-1}\} \]

Recall the involution of \( S^3 \) sending \( A \) to \(-A^{-1}\):

\[ \iota : S^3 \to S^3 \]

\[ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \to \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} \]

Hence, if \( A \in \mathbb{C}^3_+ \), then \(-A^{-1} \in \mathbb{C}^3_- \).

Now construct imbeddings \( \varphi, \psi \):

\[ \varphi : (S^3)^2 \to (S^3)^2 \]

\[ \psi : (S^3)^2 \to (S^3)^2 \]

Given by:

\[ \varphi(x_1, x_2) = (\omega(x_1, -\gamma_g(x_2)^{-1}), x_1) \]

\[ \psi(z_1, z_2) = (\omega_1(z_1, -\gamma_g(z_2)^{-1})) \]

Then we conclude that:

\[ \text{im}(\varphi) = \{(y_1, y_2) \in (S^3)^2 | \gamma_g(y_2) \in \mathbb{C}^3_+ \} \]

\[ \text{im}(\psi) = \{(y_1, y_2) \in (S^3)^2 | \gamma_g(y_2) \in \mathbb{C}^3_- \} \]

Consider the diffeomorphism:

\[ \lambda : \mathbb{R}^1_+ \times R^g_+ \to 2N^1_+ \times R^g_+ \]

\[ \lambda(x_1, x_2) = (\omega(x_1, -\gamma_g(x_2)^{-1}), p_1 \gamma^{-1}(x_2)) \]

Then we get the following commutative diagram:

\[ \begin{array}{ccc}
R^1_+ \times R^g_+ & \xrightarrow{\lambda} & 2N^1_+ \times R^g_+ \\
\text{im } \varphi \cap \text{im } \psi & \downarrow & \downarrow \\
R^1_+ \times S^2 \times R^g_+ & \xrightarrow{\lambda} & 2N^1_+ \times R^g_+ \\
\text{im } \varphi \cap \text{im } \psi & \downarrow & \downarrow \\
R^1_+ \times S^2 \times R^g_+ & \xrightarrow{\lambda} & 2N^1_+ \times R^g_+ \\
\end{array} \]

Hence:

\[ R^{g+1}_- = R^1_+ \times N^g_+ \cup N^1_+ \times R^g_- \]

Furthermore, since \( A = -A^{-1} \) for \( A \in S^3 \), we can write:

\[ \lambda(x_1, x_2) = (\omega_1(x_1, \gamma_g(x_2)), p_1 \gamma^{-1}(x_2)) \]
\( \lambda \) is the composition of:

\[
R^1_+ \times N^E_+ \xrightarrow{\iota + w_1^{-1}} R^1_+ \times S^2 \times R^E \xrightarrow{\iota + 1} \partial N_+^1 \times H^E_+.
\]

We can describe the gluing by the diagram:

\[
\begin{array}{ccc}
R^1_+ \times S^2 \times R^E & \xrightarrow{\iota + 1} & \partial N_+^1 \times H^E_+ \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
\partial N_+^1 \times H^E_+ & \xrightarrow{\iota + 1} & \partial N_+^1 \times H^E_+ \\
\end{array}
\]

where \( \tilde{w}_g \) and \( \tilde{w}_1 \) are \( w_g \) and \( w_1 \) followed by inclusions.

We have a principal fibre bundle with fibre \( SO_3 \):

\[
\begin{array}{ccc}
\tilde{g}^E & \longrightarrow & \tilde{g}^E \\
\end{array}
\]

We have previously shown that \( \tilde{R}_+^1 \) is a point. Hence, \( R^1_+ \) is homeomorphic to \( SO_3 \).

The Mayer–Vietoris sequence yields:

\[
\cdots \longrightarrow H_i(R^1_+ \times S^2 \times R^E) \xrightarrow{\lambda_i} \Pi_i(R^1_+ \times N^E_+ \oplus H_i(N^1_+ \times R^E)) \longrightarrow \cdots
\]

\[
\longrightarrow \quad \xleftarrow{\lambda_{i-1}} \quad H_i(-1)(R^1_+ \times S^2 \times R^E) \quad \xrightarrow{\beta} \quad \Pi_i(R^E_{i+1}) \quad \longrightarrow \cdots
\]

We conclude that:

\[
H_*(R^1_+ \times S^2 \times R^E) \cong H_*(\tilde{g}^E) \oplus [H_2(S^2) \oplus H_{r-1}(\tilde{g}^E)] \oplus H_{r-2}(S^3 \times R^E)
\]

\[
H_1(R^1_+ \times N^E_+ \cong H_1(N^E_+) \oplus H_{r-2}(\tilde{g}^E)
\]

\[
H_*(N^1_+ \times R^E) \cong H_*(\tilde{g}^E) \oplus (H_2(N^1_+) \oplus H_{r-2}(\tilde{g}^E)) \oplus (H_3(N^1_+) \oplus H_{r-3}(\tilde{g}^E)).
\]

We can easily check that:

\[
\lambda^*(\alpha, \beta, \gamma) = (\tilde{w}_g(\alpha, \tilde{w}_g(\beta, \tilde{w}_g(\gamma, \alpha)) \oplus \iota + 1)(\alpha, \beta, \gamma)
\]

where \( \tilde{w}_g \) denotes the map induced in \( r \)-th homology. We deduce:

\[
\ker \lambda_1 \cong \ker(\tilde{w}_g).
\]

Now, consider the exact sequence:

\[
0 \longrightarrow H_4(S^3 \times S^3, N^1_+) \longrightarrow H_3(N^1_+) \longrightarrow H_3(S^3 \times S^3) \longrightarrow \cdots
\]

Since the usual generators of \( H_3(S^3 \times S^3) \) lie in \( N^1_+ \), \( i_* \) is onto. In addition, we compute that:

\[
H_3(S^3 \times S^3, N^1_+) \cong \Pi_3(N^1_+) \quad \text{(excision)}
\]

\[
\cong H_3(\mathbb{RP}^3 \times \mathbb{RP}^3, \mathbb{RP}^3 \times S^2)
\]

\[
\cong H_3(\mathbb{RP}^3) \quad \text{(Thom isomorphism)}
\]

\[
\cong \mathbb{Z}
\]

Hence:

\[
H_3(N^1_+) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}
\]

Also, since \( i_* \) is onto, we have an exact sequence:

\[
0 \longrightarrow H_3(S^3 \times S^3, N^1_+) \longrightarrow H_3(N^1_+) \longrightarrow 0
\]

This implies that:

\[
H_3(N^1_+) \cong H_3(S^3 \times S^3, N^1_+) \cong H_3(\mathbb{RP}^3 \times D^3, \mathbb{RP}^3 \times S^2) \cong H_0(\mathbb{RP}^3)
\]

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Hence:

\[ H_2(N^g_1) = 0. \]

Similarly:

\[ H_1(N^g_1) = 0. \]

By using \( \& \) coefficients in homology, we get:

\[ \text{coker } \lambda_r \cong H_2(N^g_1, \& H_1(S^g_2, 1 \otimes 2 H_{g-3}(R^g_1)) \cong \]

Hence:

\[ H_2(L^{g+1}_1) \cong \text{coker } \lambda_r \otimes \ker \lambda_{r-1}. \]

\[ H_0(L^{g+1}_1) \cong H_0(N^g_1) \otimes \ker(\omega_{g-3}) \oplus 2 H_{g-3}(R^g_1) \oplus \ker(\omega_{g-3}). \]

\[ H_0(L^{g+1}_1) \cong H_0(N^g_1) \oplus 2 H_{g-3}(R^g_1) \oplus H_{g-3}(R^g_1)^3 \cong \]

For a fixed \( x \in L^1 \), we can define the map:

\[ \alpha : N^g_1 \rightarrow (x)^1 \times N^g_1 \rightarrow R^1 \times N^g_1 \rightarrow H_1^{g+1}. \]

It follows that:

\[ \alpha^* : H_2(L^{g+1}_1) \rightarrow H_2(R^g_1). \]

is an isomorphism. So:

\[ \alpha^* : H^2(L^{g+1}_1) \rightarrow H^2(N^g_1). \]

is an isomorphism. Let \( \mu_1, \ldots, \mu_g \) be the generators of \( H^2(N^g_1) \) induced from the obvious generators of \( (S^1)^{2g} \) by the inclusion:

\[ N^g_1 \rightarrow (S^1)^{2g}. \]

Then:

\[ \alpha^* (\mu_i) = \begin{cases} 0 & \text{for } i = 1, 2 \\ \mu_{i-2} & \text{for } 3 \leq i \leq 2g + 2 \end{cases} \]

This fact will be used later.

**Theorem 1.2:** The following inclusions are surjective:

(a) \( H_2(L^{g+1}_1) \rightarrow H_2(R^g_1) \) for \( r \leq 3g + 2 \)

(b) \( H_2(L^{g+1}_1) \rightarrow H_2(R^g_1) \) for \( r \leq 3g - 1 \).

**Proof:** (a) We first prove the case: \( g = 1 \). Since \( R^g_1 \) is \( S^2 \times \mathbb{R} \), the only relevant case is \( r = 3 \):

\[ H_2(L^{g+1}_1) \rightarrow H_2(R^g_1) \]

The obvious generators, \( S^2 \times I \) and \( I \times S^2 \), are contained in \( N^g_1 \). So they lie in \( N^g_1 \).

We then show the case: \( g = 2 \). In this case, \( R^g_1 \) is \( (S^1)^4 \). So the relevant cases are \( r = 3, 6 \):

\[ H_2(R^g_1) = \mathbb{Z}^2, \quad H_2(R^g_1) = \mathbb{Z}^6. \]

Clearly, all the generators of \( H_2(R^g_1) \) and \( 4 \) of the \( 6 \)-generators of \( H_4(R^g_1) \) lie in \( H^1 \), and, therefore, in \( N^g_1 \). It suffices to prove that given:

\[ (S^1)^2 = S^2 \times S^1 \times I \times I \rightarrow (S^1)^4 \]

and the induced homology homomorphism:

\[ H_2((S^1)^4) \rightarrow H_2((S^1)^4) \]

the image of the generator lies in the image of \( H_2(N^g_1) \). Consider the commutative diagram:

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So it suffices to show that the following is zero:

\[ (i \times 1)_*: \; H_2(R^1_+ \times D_3 R^1_- \times S^2) \longrightarrow H_2(R^2_+ \times D_3 R^2_- \times S^2) \]

Since \( R^1_- \) is \( \mathbb{R}P^3 \) and:

\[ H_2(R^2_- \times S^2) \longrightarrow H_3(R^2_- \times \mathbb{R}^3) \]

is onto, the long exact homology sequences give:

\[
\begin{array}{ccc}
0 & \longrightarrow & H_3(R^1_- \times D_3 R^1_- \times S^2) \\
& & \downarrow \phi \\
0 & \longrightarrow & H_3(R^2_- \times D_3 R^2_- \times S^2)
\end{array}
\]

\[
\begin{array}{ccc}
& & \downarrow (i \times 1)_* \\
0 & \longrightarrow & H_3(R^1_- \times D_3 R^1_- \times S^2) \\
& & \downarrow \phi
\end{array}
\]

So it suffices to prove the right hand vertical map is zero. In turn, it suffices to prove that:

\[ i_* : H_2(R^1_-) \longrightarrow H_3(R^2_-) \]

is zero. This map factors through the inclusion:

\[ i_* : H_2(R^1_-) \longrightarrow H_2(S^2 \times S^2) \]

Recall that \( R^1_- \) is \( \mathbb{R}P^3 \) and:

\[
R^1_- = ((A, B) \in S^3 \times S^3 \mid ABA^{-1}B^{-1} = -I)
\]

\[
= ((A, B) \in S^3 \times S^3 \mid ABA^{-1}B^{-1} = -I, \; \text{tr}(A) = 0, \; \text{tr}(B) = 0) \subset S^3 \times S^3 \times S^3 \times S^3.
\]

Therefore, \( i_* = 0 \).
We now assume that \( g \geq 3 \). By induction, we assume that (a) is proved \(-1\). Let \( a \neq g \). Consider a standard generator of \( H_{3s}(\mathbb{Z}^3)^{2g} \):

\[ \Sigma = \Sigma_1 \ast \cdots \ast \Sigma_g \]

where each \( \Sigma_i \) is either \( S^3 \) or \( \{1\} \) and exactly \( s \) such factors of \( \Sigma \) are \( S^3 \).

solve: \( \Sigma_{2k+1} \ast \Sigma_{2k+2} = S^3 \ast \{1\} \) or \( \{1\} \ast S^3 \) for some integer \( k \) with \( a \leq k \leq g - 1 \).

For simplicity, we assume that \( k = 0 \). By assumption:

\[ \Sigma \text{ lies in } H_3(\mathbb{Z}^3) \otimes H_{3s-3}(\mathbb{Z}^3)^{2g-2}. \]

By induction, therefore:

\[ \Sigma \text{ lies in the image of } H_3(\mathbb{Z}^3) \otimes H_{3s-3}(\mathbb{Z}^3)^{2g-2}. \]

Since:

\[ N_+^k = \Gamma_0^{-1}(D_+), \]

let:

\[ N_+^k(\varepsilon) = \Gamma_0^{-1}(D_+ (\varepsilon)) \]

where:

\[ D_+ (\varepsilon) = \{ A \in S^3 | \text{tr}(A) \leq 2 - \varepsilon \} \]

where \( 0 < \varepsilon < 2 \). \( N_+^k \) can be deformation retracted onto \( N_+^k(\varepsilon) \) and for some small positive \( \varepsilon \), \( N_+^k(\varepsilon) \) is contained in a regular neighborhood of \( N_+^k \). Since:

\[ \Sigma = (I \ast \{1\} \ast \Sigma_2 \ast \cdots \ast \Sigma_2 \ast \{1\} \ast \Sigma_g), \]

the image of \( \Sigma \) deformation retracts into \( N_+^k(\varepsilon) \). Hence, \( \Sigma \) comes from \( H_{3s}(\mathbb{Z}^3)^{2g} \).

We now prove (b). Suppose \( a = g - 1 \). Let \( \Sigma \) be a generator of \( H_{3s}(\mathbb{Z}^3)^{2g} \). Then, for some \( k \), \( \Sigma = \Sigma_1 \ast \cdots \ast \Sigma_g \). Suppose \( k = 0 \).

Then \( \Sigma = (I \ast \{1\} \ast \Sigma_2 \ast \cdots \ast \Sigma_g) \) is homological to \( \Sigma' = x_0 \ast \Sigma_3 \ast \cdots \ast \Sigma_g \) for \( x_0 \in \mathbb{Z}^3 \). Consider the imbedding:
\[ \varphi : (S^3)^{3g-3} \longrightarrow (S^3)^2 \times (S^3)^2g-3 \]

with \( \varphi(y) = (x_0, y) \). Then:

\[ \varphi(x_0^{3g-1}) \in x_0. \]

Hence, \( x_0 \) lies in the image of \( \varphi \) and the following commutes:

\[ \begin{array}{ccc}
H_{3g}( (S^3)^{3g-3} ) & \xrightarrow{\varphi_*} & H_{3g}( (S^3)^2g ) \\
\downarrow i_* & & \downarrow i_* \\
H_{3g}( N_+^{3g-1} ) & \xrightarrow{\varphi_*} & H_{3g}( N_-^g ) 
\end{array} \]

Consequently, (a) implies (b). \( \square \)

We introduce the following notation:

\[ h_r^g = \dim H_r( N_+^g ) \]
\[ n_r^g = \dim H_r( (S^3)^2g ) = \begin{cases} 2g & \text{if } r = 0(3) \\ r/3 & \text{if } r \not\equiv 0(3) . \end{cases} \]

**Theorem 1.3:** If \( r \not\leq 3g + 1 \), we have:

\[ h_r^{g+1} = h_r^{g+2} + 2h_{r-3}^g + n_r^g - n_{r-4}^g . \]

**Proof:** Since: \( H_r( N_+^g ) \longrightarrow H_r( (S^3)^2g ) \) is onto for \( r \neq 3g + 2 \):

\[ \begin{aligned}
H_r( N_+^g ) & \longrightarrow H_r( (S^3)^2g ) \\
\text{exc} & \longrightarrow H_r( (S^3)^2g ) \oplus H_{r+1}( N_-^g N_{-1}^g ) \\
\text{Thom} & \longrightarrow H_r( (S^3)^2g ) \oplus H_{r-1}( N_-^g ) , \quad \text{(since } N_-^g \cong N_+^g \times S^1) \\
\end{aligned} \]

when \( r \neq 3g + 1 \). Since:

\[ H_r( N_+^g ) \longrightarrow H_r( (S^3)^2g ) \]

is onto when \( r \neq 3g - 1 \), we conclude that:

\[ H_r( N_+^g ) = H_r( (S^3)^2g ) \oplus H_{r+1}( N_-^g N_{-1}^g ) \]

when \( r \leq 3g - 2 \).

Hence:

\[ \dim H_r( N_+^g ) = h_r^g + m_r^g \]
\[ \dim H_{r+1}( N_-^g N_{-1}^g ) = h_{r+1}^g - m_r^g . \]

From the discussion preceding Theorem 1.2:

\[ h_r^{g+1} = (h_r^{g+2} + m_r^g) + 2h_{r-3}^g + (n_r^g - n_{r-4}^g) . \]

**Theorem 1.4:**

\[ \begin{aligned}
& h_r^g = \begin{cases} \left[ \frac{s}{3} \right] & \text{if } r = 2s, \quad s \leq \left[ \frac{2g}{2} \right] - 1 \\
\frac{(s-1)/3}{2j+1} & \text{if } r = 2s + 1, \quad s \leq \left[ \frac{2g-1}{2} \right] \\
\end{cases} \\
\end{aligned} \]

(a) \( h_r^g = \)

\[ \begin{aligned}
& \begin{cases} \left[ \frac{s}{3} \right] & \text{if } r = 2s, \quad s \leq \left[ \frac{2g}{2} \right] - 1 \\
\frac{(s-1)/3}{2j+1} & \text{if } r = 2s + 1, \quad s \leq \left[ \frac{2g-1}{2} \right] \\
\end{cases} \\
\end{aligned} \]

(b) \( h_r^g = \)

\[ \begin{aligned}
& \begin{cases} \left[ \frac{s}{3} \right] & \text{if } r = 2s, \quad s \leq \left[ \frac{2g}{2} \right] - 1 \\
\frac{(s-1)/3}{2j+1} & \text{if } r = 2s + 1, \quad s \leq \left[ \frac{2g-1}{2} \right] \\
\end{cases} \\
\end{aligned} \]

**Proof:** The proof is by induction on \( g \). For \( g = 1 \), \( h_r^1 \) is \( 0 \) and (a) and (b) hold. By Theorem 1.3, for \( r \neq 3g + 1 \):
\[ h_{r+1}^g = h_r^g + 2 h_{r-1}^g + h_{r-3}^g + 2 h_{r-4}^g - h_{r-5}^g \]

after the appropriate restriction on \( m \), we have:

\[ h_{4m+1}^g = h_{4m}^g + 2 h_{4m-1}^g + h_{4m-3}^g + 2 h_{4m-4}^g - 2 h_{4m-2}^g \]

\[ h_{4m+2}^g = h_{4m+1}^g + 2 h_{4m+2}^g + h_{4m-3}^g + 2 h_{4m-1}^g + 2 h_{4m-4}^g - 2 h_{4m-2}^g \]

\[ h_{4m+3}^g = h_{4m+2}^g + 2 h_{4m+3}^g + h_{4m-3}^g + 2 h_{4m-1}^g + 2 h_{4m-2}^g - 2 h_{4m-3}^g \]

\[ h_{4m+4}^g = h_{4m+3}^g + 2 h_{4m+4}^g + h_{4m-1}^g + h_{4m-2}^g - 2 h_{4m-3}^g \]

\[ h_{4m+5}^g = h_{4m+4}^g + 2 h_{4m+5}^g + h_{4m-1}^g + 2 h_{4m-2}^g - h_{4m-3}^g \]

To prove formula (a) by induction on \( g \). We give the proof when \( r = 3m \). The other cases are proved similarly. From above:

\[ h_{3m}^g = 2 h_{3m-1}^g + 2 h_{3m-2}^g + h_{3m-3}^g + \left( \frac{2g}{2m} \right) \]

sum:

\[ 3m = \left( \frac{3g + 1}{2} \right) - 1 \]

This implies:

\[ 3m - 2 \leq 3m - 1 \leq \left( \frac{3g}{2} \right) - 1 \]

\[ 3m - 2 \leq \left( \frac{3g - 3}{2} \right) \]

Now, by induction, we have:

\[ h_{3m}^g = \sum_{j=0}^{m-1} \left( \frac{2g}{2j+1} \right) + 2 \sum_{j=0}^{m-1} \left( \frac{2g}{2j+3} \right) \]

\[ = \sum_{j=0}^{m} \left( 2g \cdot \left( \frac{2j+1}{2} \right) \right) + \sum_{j=0}^{m} \left( 2g \cdot \left( \frac{2j+3}{2} \right) \right) \]

\[ = \sum_{j=0}^{m} \left( \frac{2g}{2} \right) + \sum_{j=0}^{m} \left( 2g \right) \]

\[ = \sum_{j=0}^{m} \left( 2g \cdot \left( \frac{2j}{2j+1} \right) \right) + \sum_{j=0}^{m} \left( 2g \cdot \left( \frac{2j}{2j+3} \right) \right) \]

\[ = \sum_{j=0}^{m} \left( 2g \cdot \left( \frac{2j}{2j+1} \right) \right) + \sum_{j=0}^{m} \left( 2g \cdot \left( \frac{2j}{2j+3} \right) \right) \]

\[ = \sum_{j=0}^{m} \left( 2(gj+1) \right) \]

\[ = \sum_{j=0}^{m} \left( \frac{2(gj+1)}{2} \right) \]

The dimension of \( H^g \) is 6g - 3.

Hence, Poincare duality gives (b). \( \Box \)

**Corollary 1.5:** \( h_{sg-1}^g = h_{sg-2}^g = 0 \) and \( h_1^g = 0 \).

**Proof:** The first equality comes from (b) of Theorem 1.4. If \( g \) is odd:

\[ 3g - 2 = 2 \left( \frac{3(g-1)}{2} \right) + 1 \]

If \( g \) is even:

\[ 3g - 2 = 2 \left( \frac{3g}{2} - 1 \right) \]

Hence, Theorem 1.4(a) gives \( h_{sg-2}^g = 0 \). Likewise, \( h_1^g = 0 \). \( \Box \)

Consider the fibration:

\[ \text{SO}_3 \longrightarrow H^g \text{ spin} \]

\[ \downarrow p \]

\[ H^g \text{ spin} \]

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**Theorem 1.6**: If \( r \leq 3g - 2 \), then:

\[
P^\#: H^r(\tilde{\mathcal{X}}_g; \mathfrak{g}) \longrightarrow H^r(\mathcal{X}_g; \mathfrak{g})
\]

is an injection.

**Proof**: Since \( \mathcal{S}_3 = \mathbb{R}^3 \) is a \( \mathfrak{g} \)-homology 3-sphere, there is a Thom-Gysin exact sequence:

\[
0 \longrightarrow H^r(\tilde{\mathcal{X}}_g; \mathfrak{g}) \xrightarrow{\partial^r} H^r(\mathcal{X}_g; \mathfrak{g}) \xrightarrow{\rho^r} H^{r-3}(\tilde{\mathcal{X}}_g; \mathfrak{g}) \xrightarrow{\rho^r} H^{r-1}(\tilde{\mathcal{X}}_g; \mathfrak{g}) \longrightarrow \cdots
\]

where all cohomologies are taken with \( \mathfrak{g} \)-coefficients and:

\[
x \in H^r(\tilde{\mathcal{X}}_g; \mathfrak{g})
\]

also make the convention that the rest of the homologies in this proof are taken with \( \mathfrak{g} \)-coefficients.

Since:

\[
H^r(\mathcal{X}_g; \mathfrak{g}) \cong \text{Hom}(H_r(\mathcal{X}_g; \mathfrak{g}), \mathfrak{g})
\]

it suffices to prove that:

\[
P^\#: H^r(\tilde{\mathcal{X}}_g; \mathfrak{g}) \longrightarrow H^r(\mathcal{X}_g; \mathfrak{g})
\]

is onto for \( r \leq 3g - 2 \). In turn, it is enough to prove that the map:

\[
u_X : H^{r-3}(\tilde{\mathcal{X}}_g; \mathfrak{g}) \longrightarrow H^{r+1}(\tilde{\mathcal{X}}_g; \mathfrak{g})
\]

is injective for \( r \leq 3g - 2 \). We prove this by descending induction on \( r \), starting with the cases \( r = 3g - 2 \) and \( r = 3g - 3 \). Consider:

\[
H^{3g-3}(\tilde{\mathcal{X}}_g) \xrightarrow{\nu_X} H^{3g-1}(\tilde{\mathcal{X}}_g) \xrightarrow{\rho^3} H^{3g-1}(\mathcal{X}_g)
\]

By Corollary 1.5, the two outside groups are zero, so \( \nu_X \) is an isomorphism.

Similarly:

\[
u_X : H^{3g-3}(\tilde{\mathcal{X}}_g) \longrightarrow H^{3g-1}(\tilde{\mathcal{X}}_g)
\]

is onto. Since \( \tilde{\mathcal{X}}_g \) is a Kähler manifold [N1], there is \( \omega \in H^2(\mathcal{X}_g) \) such that:

\[
u_X : H^2(\mathcal{X}_g) \longrightarrow H^{3g-3}(\tilde{\mathcal{X}}_g)
\]

is injective for \( r = 3g - 4 \). Hence:

\[
H^2(\mathcal{X}_g) \xrightarrow{\nu_X} H^2(\mathcal{X}_g) \xrightarrow{\rho^2} H^{2g-2}(\mathcal{X}_g)
\]

is injective and:

\[
\dim H^2(\mathcal{X}_g) = \dim H^{2g-2}(\mathcal{X}_g)
\]

Hence:

\[
u_X : H^{2g-2}(\mathcal{X}_g) \longrightarrow H^{2g-2}(\mathcal{X}_g)
\]

is an isomorphism.

Now assume \( r \leq 3g - 4 \) and:

\[
u_X : H^{r-3}(\tilde{\mathcal{X}}_g) \longrightarrow H^{r+1}(\tilde{\mathcal{X}}_g)
\]

is injective for all \( r \leq 3g - 2 \). Consider:

\[
u_{uX} : H^{r-3}(\tilde{\mathcal{X}}_g) \longrightarrow H^{r+1}(\tilde{\mathcal{X}}_g)
\]

It can be written as compositions in two different ways:

\[
H^{r-3}(\tilde{\mathcal{X}}_g) \xrightarrow{\nu_X} H^{r+1}(\tilde{\mathcal{X}}_g) \xrightarrow{\rho^r} H^{r+1}(\tilde{\mathcal{X}}_g)
\]

\[
H^{r-3}(\tilde{\mathcal{X}}_g) \xrightarrow{\nu_{uX}} H^{r+1}(\tilde{\mathcal{X}}_g) \xrightarrow{\rho^r} H^{r+1}(\tilde{\mathcal{X}}_g)
\]

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the second factorization, both homomorphisms are injective by the
perties of $w$ and by induction. Hence:

$$\eta^{-1}(\bar{R}_r) \to H^{r+1}(\bar{R}_r')$$ is injective. \qed

**Corollary 1.7:** $p^* : H^r(\bar{R}_r') \to H^r(\bar{R}_r)$ is an isomorphism for $r \neq 3$ and 2.

**Proof:** Consider again the Thom-Gysin sequence:

$$\cdots \to H^{r+1}(\bar{R}_r') \to H^r(\bar{R}_r') \to H^r(\bar{R}_r) \to H^r(\bar{R}_r') \to \cdots$$

course, the first term is zero. Hence, $p^*$ is injective for $r \neq 3$. On the other hand, by Theorem 1.6, $p^*$ is surjective for $r \neq 3$ (provided $3q - 2$).

**Corollary 1.8:** For $r \leq 3q - 2$:

$$\dim H_r(\bar{R}_r') = \frac{[r/4]}{4} \sum_{j=0}^{[r/4]} b^r_j.$$ \qed

**Proof:** By Theorem 1.6, for $r \leq 3q - 2$, we have a short exact sequence:

$$0 \to H^{r-1}(\bar{R}_r') \to H^r(\bar{R}_r') \to H^r(\bar{R}_r) \to 0.$$ \vspace{1em}

$$\eta_r^g = \dim H^r(\bar{R}_r').$$ \vspace{1em}

$$\eta_r^g = \eta_r^{g-a} + \eta_r^g.$$ \vspace{1em}

s gives the required formula. \qed

Consider the maps:

$$i_+ : N_r^g \to N_{r+1}^g$$ $i_+(w) = (w, 1, 1)$

$$i_- : N_r^g \to N_{r+1}^g$$ $i_-(w) = (w, 1, 1)$

$$j : S^2 \to \bar{R}_r^g \times S^2$$ $j(w) = ((w, 1, \ldots, 1), w).$

**Lemma 1.9:** The maps $\bar{w}_g \cdot j$ and $i_-$ induce isomorphisms on $\mathbb{H}^2$. If $g \geq 2$, $i_-$ also induces an isomorphism.

**Proof:** Since, by Theorem 1.2:

$$i_r^* : H_2^g(N_4^g) \to H_2^g((S^3)^g)$$

is surjective:

$$i_r^* : H_2^g((S^3)^g) \to H_2^g(N_4^g)$$

is injective. Hence, Mayer-Vietoris gives:

$$0 \to H_2^g(N_4^g) \times H_2^g(N_4^g) \to H_2^g(N_4^g) \times H_2^g(N_4^g) \to 0.$$ \vspace{1em}

The map is induced by:

$$\times \mapsto (\bar{w}_g(w), \bar{w}_g(w)).$$ \vspace{1em}

Since the image of:

$$\bar{w}_g : H_2^g(N_4^g) \to H_2^g(S^3_g \times S^2)$$

is the kernel of $j^*$, $(\bar{w}_g \cdot j)^*g$ is an isomorphism.

Consider the diagram:

$$S^2 \xrightarrow{j} R_r^g \times S^2 \xrightarrow{\bar{w}_g} N_r^g \xrightarrow{\alpha} R_r^g$$

$$\downarrow 1 \quad \downarrow i_- \quad \downarrow i_+ \quad \downarrow i_-$$

$$S^2 \xrightarrow{j} R_{r+1}^g \times S^2 \xrightarrow{\bar{w}_g+1} N_{r+1}^g \xrightarrow{\alpha} R_{r+1}^g.$$
commutes except on the middle square which is homotopy commutative, i.e.:

\[
(\bar{w}_g \circ -1) \circ i_+ \circ \bar{w}_g : H_\infty^g \times S^2 \longrightarrow H_{g+1} \times S^2
\]

homotopic to \( i_+ \). (This follows from the proof of Theorem 1.2(a) by

\[
\mathbb{B}^g \times \mathbb{D}^3 \longrightarrow \mathbb{B}^{g+1} \times \mathbb{D}^3
\]

the inclusion:

\[
\mathbb{B}^g \times \mathbb{D}^3 \longrightarrow \mathbb{B}^g \times \mathbb{D}^3
\]

be taken to fix the second factor. Hence, it restricts to the desired

cology on \( H_\infty^g \times S^2 \).

Hence, we have the commutative diagram:

\[
\begin{array}{ccc}
H^s(S^3) & \xrightarrow{\bar{w}_g \circ j} & H^s(H^g_\infty) \\
\alpha \downarrow & & \alpha \downarrow \\
H^s(S^2) & \xrightarrow{\bar{w}_g \circ i_+} & H^s(H^g_\infty) \\
\end{array}
\]

\( i_+ \) and \( i_- \) induce isomorphisms on \( H^2 \). \( \square \)

Let \( \lambda \) be the generator of \( H^2(H^g_\infty) \). Also, let \( \lambda \) denote the generators

\( H^2(H^g_\infty) \) and \( H^2(H^g_\infty) \) for \( g \geq 2 \) induced by the isomorphism of the above

\[
\mu_1, \ldots, \mu_g \text{ be the elements of } H^2(H^g_\infty) \text{ and } H^2(H^g_\infty) \text{ induced by inclusions:}
\]

\[
\mu_i \text{ for } i \leq 2g,
\]

\[
\mu_{i-2g} \text{ for } i = 2g + 1, 2g + 2.
\]

We have natural inclusions:

\[
H^g_+ \times H^g_+ \longrightarrow (S^2)^{2g} \times (S^2)^3 = (S^3)^{2g+1}.
\]

As in the proof of Theorem 1.2(a), Case 2, we can deform this map into a map:

\[
\mathbb{B}^g \times H^g_+ \longrightarrow \mathbb{B}^{g+1}.
\]

Lemma 1.10: (1) \( \iota_*(\lambda^p) = \lambda^p \otimes 1 + p\lambda^{p-1} \otimes \lambda \)

(2) \( \iota_*(\mu_i) = \begin{cases} 
\mu_i \otimes 1 & \text{for } i \leq 2g, \\
1 \otimes \mu_{i-2g} & \text{for } i = 2g + 1, 2g + 2.
\end{cases} \)

Proof: Part (2) is trivial. To prove part (1), we begin by observing that the
discussion preceding Theorem 5.1 implies:

\[
H_1(N^g_\infty) \otimes H_1(H^g_\infty) = 0 \text{ for } g \geq 1.
\]

Hence, we compute that:

\[
H^2(N^g_\infty \times N^g_\infty) \cong H^2(N^g_\infty) \otimes H^2(N^g_\infty) = 0.
\]

We may write the pullback of \( \lambda \) as follows:

\[
\iota^*(\lambda) = a(\lambda \otimes 1) \oplus b(1 \otimes \lambda).
\]
Now we have the following diagrams:

\[
\begin{array}{ccc}
N^i_+ & \xrightarrow{i} & N^i_+ \times N^j_+ \\
\downarrow & & \downarrow i^* \\
\mathbb{R}^d & \xrightarrow{j} & \mathbb{R}^d \times \mathbb{R}^j
\end{array}
\]

\[
i(x) = (x,1)
\]

\[
j(x) = (1,x)
\]

Choosing the deformation to \( \lambda \) carefully, we may ensure that these programs commute up to homotopy. Hence, we have commutative diagrams:

\[
\begin{array}{ccc}
H^2(N^i_+) & \xrightarrow{i^*} & H^2(N^i_+ \times N^j_+) \\
\downarrow & & \downarrow i^* \\
H^2(N^i_-) & \xrightarrow{j^*} & H^2(N^j_+)
\end{array}
\]

\[
H^2(N^i_+) & \xrightarrow{j^*} & H^2(N^i_+ \times N^j_+) \\
\downarrow & & \downarrow j^* \\
H^2(N^j_-) & \xrightarrow{i^*} & H^2(N^j_-)
\]

By definition, \( i^*(\lambda) = \lambda \) and \( j^*(\lambda) = \lambda \). Hence, we conclude that:

\[
\lambda = i^*(\lambda) = (i')^*j^*(\lambda) = a\lambda
\]

\[
\lambda = j^*(\lambda) = (j')^*i^*(\lambda) = b\lambda
\]

This implies that \( a = 1 \) and \( b = 1 \):

\[
\lambda = i^*(\lambda) = (i')^*j^*(\lambda) = \lambda
\]

\[
\lambda = j^*(\lambda) = (j')^*i^*(\lambda) = \lambda
\]

Since \( H^*(N^i_+) = 0 \), by the proof of Theorem 1.3, we conclude, by induction on \( p \), that:

\[
\lambda^p = \lambda^p \otimes 1 \otimes 1 \otimes \cdots \otimes \lambda
\]

**Proposition 1.21:** The elements \( \lambda^p q_{i1} \cdots q_{ir} \) in \( H^*(N^j_+) \), where \( p \geq 0 \), \( 1 \leq q_{i1} < \cdots < q_{ir} \leq 2g \), and \( p + r \leq g \), are linearly independent.

**Proof:** The proof is by induction on \( r \). Assume that the result holds for \( r - 1 \geq 1 \). Recall:

\[
\lambda \in N^j_+ \times N^j_+ \xrightarrow{\lambda} N^j_+
\]

Let \( \Sigma \) be a linear combination of elements of the form \( \lambda^p q_{i1} \cdots q_{ir} \). Assume \( \Sigma = 0 \). Look at the component of \( \lambda^p(\Sigma) \) in:

\[
H^*(N^j_+) \otimes H^*(N^j_+)
\]

By induction on \( r \) and Lemma 1.10, we conclude that the only terms in \( \Sigma \) which can have nonzero coefficients are of one of the following forms:

(a) the ones with \( p = 0 \)

(b) the ones containing \( \mu_{2g-1} \) or \( \mu_{2g} \)
We can find an automorphism of $N_{+}^{g-1}$ which takes $\mu_{s_{g-1}}$ and $\mu_{s}$ to any $\mu_{s_i}$ for all $i$, $1 \leq i \leq g$. Hence, we can replace (b) by:

(b') the ones containing at least one of $\mu_{s_{g-1}}, \mu_{s_i}$ for all $i$, $1 \leq i \leq g$.

Hence, all terms of $\Sigma$ satisfy (a) and are in the form:

$$\mu_{q_1} \cdots \mu_{q_r} \quad r \leq g.$$ 

At the inclusion induces an injection:

$$H^k((S^3)^{2g}) \longrightarrow H^k(N_{+}^{g})$$

or $r \leq 3g + 2$ and such elements are linearly independent in $H^k((S^3)^{2g})$, since they are linearly independent in $H^k(N_{+}^{g})$. The case of genus one is true by inspection. \(\square\)

Corollary 1.12: The element $\lambda$ generates $H^2(N_{+}^{g})$. The element $\lambda^2$ generates $H^3(N_{+}^{g})$.

Proof: From Theorem 1.2 and the long exact sequence for a pair, we deduce he short exact sequence:

$$0 \longrightarrow H^r(N_{+}^{g}) \longrightarrow H^r(N_{+}^{g}) \longrightarrow H^r((S^3)^{2g}) \longrightarrow 0$$

or $r \leq 3g + 1$. Hence, we deduce that:

$$H^2(N_{+}^{g}) \cong H^2((S^3)^{2g}, N_{+}^{g})$$

$$\cong H^2(N_{+}^{g}, 2N_{+}^{g}) \quad (\text{excision})$$

$$\cong H^2(N_{+}^{g}) \quad (\text{Thom}).$$

Consequently, $H^2(N_{+}^{g})$ has rank one and the result follows from Proposition 1.11.

In a similar manner, we deduce that:

$$H^3(N_{+}^{g}) \cong H^3(R_{+}^{g}).$$

By Theorem 1.4, we conclude that the rank of $H^4(N_{+}^{g})$ is zero in genus one and one in genus $g = 2$. Again, the result follows from Proposition 1.11. \(\square\)

Proposition 1.13: The elements $\lambda^r \mu_{q_1} \cdots \mu_{q_r}$, $p > 0$, $1 \leq q_1 < \cdots < q_r \leq 2g$, $p + r 

\leq g - 1$ are linearly independent in $H^k(R_{+}^{g})$.

Proof: This is obvious for $g = 1$. Let $\Sigma$ be a linear combination of elements as above such that $\Sigma = 0$. As above, the only terms with non-zero coefficients are in the form (b')'. But $p + r < g - 1$ implies that there are no such terms. \(\square\)

Corollary 1.14: The elements $\lambda^r \mu_{q_1} \cdots \mu_{q_r}$, $p > 0$, $1 \leq q_1 < \cdots < q_r \leq 2g$, $p + r 

\leq g - 1$ generate $H^k(R_{-})$.

Proof: By Theorem 1.4, we know the rank $h_{+}^{k}$ of $H^k(R_{+}^{g})$ satisfies:

$$h_{r}^{k} \leq \# \{ \lambda^r \mu_{q_1} \cdots \mu_{q_r} \mid p > 0, 1 \leq q_1 < \cdots < q_r \leq 2g, p + r \leq g - 1 \}.$$ 

Hence, the result follows. \(\square\)

Corollary 1.15: The element $\lambda$ generates $H^2(R_{+}^{g})$. The element $\lambda^2$ generates $H^3(R_{+}^{g})$. 

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Both claims are obvious for $g = 1$. The first claim follows from Corollary 1.14 for $g = 2$. The second claim follows for $g = 3$. On the other hand, by Theorem 1.4:

$$H^*(R^g) = 0.$$ 

ce, the second claim is also true when $g = 2$. \[\Box\]

**Corollary 1.16:** For $r \geq 3g - 2$, $H^r(R_n)$ is generated by $\lambda, \mu_1, \ldots, \mu_2g$ and (as a cohomology ring).

**Proof:** For $r \geq 3g - 2$, by Theorem 1.6, we have the exact sequence:

$$0 \to H^{r-2}(R_n) \xrightarrow{\partial} H^r(R_n) \xrightarrow{p^*} H^r(R^g) \to 0,$$

where $x \in H^*(R^g)$. Hence, the proof follows by induction on $r$. \[\Box\]

Since $R_n$ is a nonsingular projective variety of complex dimension $- 3$ [Ni], it follows from the Hard Lefschetz Theorem ([GH], page 122) that we have a stronger result:

**Corollary 1.17:** $H^*(R_n)$ is generated by $\lambda, \mu_1, \ldots, \mu_2g$ and $x$ (as a ring).

### The action of the Torelli group

### The action of the Torelli group

Recall that every base point preserving, homomorphism, $f: (R^g, 0) \to (0,0)$ induces an homomorphism of $R^g$:

$$f^*: R^g \to R^g, \quad p \mapsto p + f^*p,$$

where $f^*$ is the induced homomorphism on $H_i(R^g, 0)$. If in addition, $f$ is orientation preserving, then $f^*$ preserves the boundary of $R^g$ and we have a commutative diagram:

$$\begin{array}{ccc}
R^g & \xrightarrow{f^*} & R^g \\
\downarrow & & \downarrow \\
S^3 & \xrightarrow{id} & S^3
\end{array}
$$

**Note:** If $f$ is orientation reversing, then we must replace the right hand vertical arrow by the inversion map, $A \mapsto A^{-1}$.

By Proposition 7, it follows that $f^*$ preserves the orientation of $R^g$.

**Note:** If $f$ is orientation reversing, this assertion is dependent on the parity of $g$.

From the previous diagram and remark, we obtain from every base point preserving, orientation preserving homomorphism, $f$, the following restrictions:

$$f^*: R^g \to R^g,$$

$$f^*: \to N^g \to N^g.$$

From the compatibility with $\partial$ and the previous remark, it follows that all these induced maps preserve orientations.

Since these maps commute with the $SO_2$ action, we obtain orientation preserving maps of orbit spaces.
\[ \tilde{\varphi} : \tilde{N}_g^R \longrightarrow \tilde{N}_g^R \]

\[ \varphi : \tilde{N}_g^R \longrightarrow \tilde{N}_g^R \]

The mapping class group \( \Gamma(F^g,0) \) is defined by:

\[ \Gamma(F^g,0) = \pi_0(\text{Homeo}^+(F^g,0)) \]

where \( \text{Homeo}^+(F^g,0) \) is the group of orientation preserving homeomorphisms of \((F^g,0)\) with the compact open topology. We denote the mapping class of \( f \) by \( f \). It is evident that the maps induced from \( f \) depend only on the mapping class of \( f \). Hence, we have well defined orientation preserving actions of \( \Gamma(F^g,0) \) on these various spaces.

The Torelli group \( T(F^g,0) \) is defined by:

\[ T(F^g,0) = \{ f \in \Gamma(F^g,0) \mid f_* = \text{id} : H_1(F^g,\mathbb{Z}) \longrightarrow H_1(F^g,\mathbb{Z}) \} \]

The above actions, of course, restrict to \( T(F^g,0) \).

(b) **Triviality on the homology of \( \tilde{N}_g^R \)**

We wish to show that the Torelli group acts trivially on the rational homology of \( \tilde{H}_g^R \). Since rational homology and cohomology are dual, we can consider rational cohomology. On the other hand, by Corollary 1.17, it suffices to consider dimensions 2, 3 and 4. We begin with dimension 3. (All homology and cohomology will be assumed to be rational.)

**Lemma 2.1:** \( T(F^g,0) \) acts trivially on \( H^3(\tilde{H}_g^R) \).

**Proof:** By Proposition 1.3.1, \( T(F^g,0) \) acts trivially on \( H^3(\tilde{H}_g^R) \). Since \( \tilde{H}_g^R \) is homotopy equivalent to \( \tilde{N}_g^R \), Theorem 1.2 implies that we have a surjection:

\[ H_2(\tilde{H}_g^R) \longrightarrow H_2(\tilde{H}_g^R) \]

On the other hand, by Theorem 1.4, the rank of \( H_2(\tilde{H}_g^R), h_2^R \), is \( 2g \). Since this is clearly the rank of \( H_2(\tilde{R}_g^R) \), the inclusion induces an isomorphism. Hence, we have an isomorphism:

\[ H_2(\tilde{R}_g^R) \stackrel{i_*}{\cong} H_2(\tilde{R}_g^R) \]

By Corollary 1.7, the natural projection induces an isomorphism (provided \( g \geq 2 \)):

\[ H_2(\tilde{R}_g^R) \longrightarrow H_2(\tilde{R}_g^R) \]

Since these identifications are compatible with the actions, we deduce that \( T(F^g,0) \) acts trivially on \( H^3(\tilde{H}_g^R) \) provided \( g \geq 2 \). The case of genus one is obvious. \( \Box \)

We now consider dimension 2.

**Lemma 2.2:** \( T(F^g,0) \) acts trivially on \( H^2(\tilde{H}_g^R) \).

**Proof:** As usual, we may assume that \( g \geq 2 \). By Corollary 1.7, it suffices to consider \( H_2(\tilde{H}_g^R) \). Tracing through the identification:

\[ H_2(\tilde{H}_g^R) \longrightarrow H_2(\tilde{H}_g^R) \]

given in the proof of Corollary 1.12, we find that this isomorphism is also compatible with the actions. (The reader should convince himself of this
assertion for the Thom isomorphism. Here we use the fact that the actions are isomorphism preserving.) Hence, we may reduce the problem to $H_4(N)$.

By Corollary 1.12, on the other hand, it suffices to consider $H_4(N)$. Again, using the identifications outlined in Proposition 1.11, we have:

$$H_4(N) \xrightarrow{\cong} H_4^*(R^k)$$

which reduces the proof to $H_4(R^k)$. But certainly $T(F^2, 0)$ acts trivially on $H_4^*(R^k)$. \qed

Finally, we consider dimension 4.

**Lemma 2.3:** $T(F^2, 0)$ acts trivially on $H_4^*(R^k)$. \begin{proof} From Corollary 1.17, Lemma 2.1 and Lemma 2.2, the proof is reduced to considering the action of $T(F^2, 0)$ on the element $x$. But $x$ is the $n$-class of the bundle:

$$SO_3 \rightarrow R^k$$

a rational orientable sphere-bundle. Since $T(F^2, 0)$ acts orientably on this bundle, we conclude that $T(F^2, 0)$ preserves $x$. \end{proof}

As observed at the beginning of this part, we can deduce the following result from Corollary 1.17, Lemma 2.1, 2.2, and 2.3.

**Corollary 2.4:** $T(F^2, 0)$ acts trivially on $H^4(R^k)$. 

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**References**


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