Cappell-Shaneson homotopy 4-spheres are standard

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4-manifolds are spaces that locally look like $\mathbb{R}^4$
We visualize 4-manifolds by handles

\[ M^4 = \]

A k-handle is just a ball \( B^4 = B^k \times B^{4-k} \) (\( k = 0, 1, 2, 3, 4 \)) attached along \( \partial B^k \times B^{4-k} = S^{k-1} \times B^{4-k} \).
2-handles

1-handle

$S^3$

$B^4$

he lives in 4-manifold!

handle slide

Cappell-Shaneson homotopy 4-spheres are standard
The rules of 4-dimensional world:

- handle slide
- canceling 3-handle
- canceling 1-handle
- surgery

-1 example of handle slide

Cappell-Shaneson homotopy 4-spheres are standard
1976 Cappell and Shaneson gave a sequence $\Sigma_m$, $m = 1, 2, \ldots$ of homotopy $S^4$'s and asked whether they are standard or exotic (some of them double cover exotic $\mathbb{RP}^4$'s which they constructed). $\Sigma_m$ is obtained surgering the circle from the mapping torus of $T^3$ with the diffeomorphism $T^3 \to T^3$ induced by the following matrix

$$A_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & m + 1 \end{pmatrix}$$

The diagram shows the process of surgering the circle $T^3 \times [0,1]$ to obtain $\Sigma_m$. The surgering is indicated by the arrow labeled $f$. This process is part of Cappell-Shaneson homotopy 4-spheres being standard.
1977 (A and Kirby) $\Sigma_0$ is obtained from $S^4$ by a Gluck construction to an imbedded 2-sphere $S^2 \subset S^4$ (a knotted copy of $S^2$) with tubular neighborhood $S^2 \times D^2$ so that

$$\Sigma_0 = (S^4 - S^2 \times D^2) \cup_f (S^2 \times D^2)$$

where the gluing map $f : S^2 \times S^1 \to S^2 \times S^1$ is the nontrivial diffeomorphism given by $f(x, y) = (\alpha(y)x, y)$, where $\alpha \in \pi_1 SO_3 = \mathbb{Z}_2$ is the nontrivial generator $\alpha : S^1 \to SO_3$

At the time we mistakenly claimed $\Sigma_0$ is $S^4$, since we overlooked checking if the gluing map is the identity or $f$. It turned out it was in fact $f$. This was first noticed by Aitchison and Rubinstein.
The knotted $S^2$ in $S^4$
Killing 3-handles of $\Sigma_0$ by turning it upside down

- 1981 (A) By turning the handlebody of $\Sigma_0$ upside down and then cancelling its 1-handles, you can get rid of all the 3-handles of $\Sigma_0$. Here is the picture:

\[ \pi_1(\Sigma_0) = \langle x, y \mid xyx = yxy, x^5 = y^4 \rangle \]
Checking $\pi_1(\Sigma_0) = 0$ (original check was lengthy, first done with the Rutgers group theory computer, then gradually simplified with some help from A.Casson)

\[
\pi_1(\Sigma_0) = \langle x, y \mid yxy = xyx, \ x^5 = y^4 \rangle
\]

\[
yxy = xyx
\]

\[
y = (yx)^{-1}x(yx)
\]

\[
y^5 = (yx)^{-1}x^5(yx) = (yx)^{-1}y^4(yx) = x^{-1}y^4x = x^{-1}x^5x = x^5 = y^4
\]

$\Rightarrow \ y = 1 \text{ and } x = 1$
$\Sigma_0$ is homeomorphic to $S^4$! Is it diffeomorphic to $S^4$?

Mazur's swindle: \( ab=1, \ ba=1 \), so \( a = a(ba)(ba)\ldots = (ab)(ab)\ldots = 1 \)
1985 (A, Kirby) Conjecture $\Sigma_0$ is possibly exotic.

1987 (Gompf) $\Sigma_0 \approx S^4$! (using the 3-handle free handlebody $\Sigma_0$, and the trivialization of $\pi_1(\Sigma_0) = \langle x, y \mid xyx = yxy, x^4 = y^5 \rangle$)

1991 (Gompf) A similar handlebody picture for $\Sigma_m$ discussed.

06/28/2009 (Gompf, Freedman, Morrison and Walker) Conjecture $\Sigma_m$ are possibly exotic when $m \neq 0$ by using modern tools from: Khovanov homology and Microsoft computers. "Man and machine thinking about the smooth 4-dimensional Poincare conjecture"

07/01/2009 (A) All $\Sigma_m \approx S^4$ (by locating cancelling 2/3-handle pairs from upsidedown view, and then identifying $\Sigma_m$ with $\Sigma_{m-1}$).

08/13/2009 (Gompf) Some more CS-spheres are standard (corresponding to some matrices other then $A_m$’s) (by using first A-Kirby paper, plus an “undoing a log-transform by fishtail” trick).
Cappell-Shaneson homotopy 4-spheres are standard
The loops $\alpha$ and $\beta$ are the unknots on the boundary.
Describing the diffeomorphism $f$

$\text{the unknot}$

$\text{isotopy}$

$\text{blow down}$

$\text{handle cancellation}$
The proof:

\[ \Sigma_m \approx \Sigma_m + \beta^{-1} \approx \Sigma_{m-1} + \alpha^{-1} \approx \Sigma_{m-1} \]

\[ \Sigma_m \approx \Sigma_{m-1} \ldots \approx \Sigma_0 \approx S^4 \]