The Dot Product

Defⁿ If $\vec{a} = (a_1, a_2, \dots, a_n)$ and $\vec{b} = (b_1, b_2, \dots, b_n)$ are two vectors, their dot product is denoted $\vec{a} \cdot \vec{b}$ and is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum_{k=1}^{n} a_k b_k .$$

Note: * For $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$,

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

* $\vec{a} \cdot \vec{b}$ is a scalar!

Length or Norm

Defⁿ If $\vec{a}$ is a vector, its length or norm is denoted by $\|\vec{a}\|$ and is defined by

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$$

Note: for a vector $\vec{a} \in \mathbb{R}^2$ $\quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$
$\vec{a} = (a_1, a_2)$ (two-dimensions)

for a vector $\vec{a} \in \mathbb{R}^3$ $\quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
$\vec{a} = (a_1, a_2, a_3)$ (three-dimensions)

for a vector $\vec{a} \in \mathbb{R}^n$ $\quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$
$\vec{a} = (a_1, a_2, \dots, a_n)$ (n-dimensions)

$\boxed{\text{Unit Vectors}}$

$\boxed{\text{Def}^{n}}$ A unit vector $\vec{u}$ is a vector whose length equals 1.

i.e. $\vec{u} \cdot \vec{u} = 1$

For example, in three-dimensional space,

$$\vec{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

is a unit vector.

Why? $\vec{u} \cdot \vec{u} = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = 1$

Hence $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = 1.$

Obtaining a unit vector along $\vec{v}$

Given a vector $\vec{v}$

$$\boxed{\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \text{ is a unit vector in the same direction as } \vec{v}.}$$

i.e., divide any non-zero vector $\vec{v}$ by its length $\|\vec{v}\|$

Example: Let $\vec{a} = (1, 2, 2)$.

then $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.$

the vector $\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ or $\dfrac{(1, 2, 2)}{3}$ is a unit vector along the same direction as $\vec{a}$.

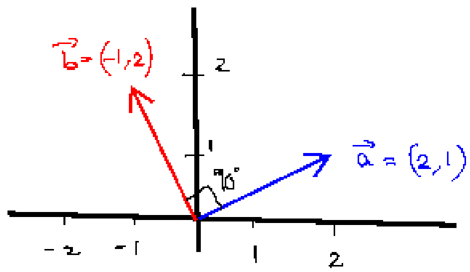
$$\boxed{\text{For any two non-zero vectors } \vec{a}, \vec{b}, \text{ the angle between } \vec{a}, \vec{b} \text{ satisfies } \cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}}$$

Note: another way of writing this: $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

What happens when $\theta = 90°$?

$$\boxed{\text{When } \vec{a} \text{ is perpendicular to } \vec{b}, \text{ the dot product } \vec{a}\cdot\vec{b} = 0}$$

For example



$\vec{a} = (2,1)$ and $\vec{b} = (-1,2)$

$\vec{a}\cdot\vec{b} = (2)(-1) + (1)(2) = 0$

$$\boxed{\text{Cauchy - Schwartz Inequality}}$$

Recall: angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ satisfies $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$.

Since $|\cos\theta| \leq 1$, $\boxed{|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|}$

Alternate expressions. $(\vec{a}\cdot\vec{b})^2 \leq (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

$\boxed{\text{Triangle Inequality}}$

If $\vec{a}$ and $\vec{b}$ are vectors, we have

$$\boxed{\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|}$$

Proof sketch: To avoid square roots let's work with $\|\vec{a}+\vec{b}\|^2 \leq \left(\|\vec{a}\| + \|\vec{b}\|\right)^2$

$$LHS = \|\vec{a}+\vec{b}\|^2 \underset{\text{using def}^n\text{ of length}}{=} (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$

$= \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a}$ (commutativity)

$$= \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$$RHS = \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$$

LHS $\leq$ RHS if $\vec{a}\cdot\vec{b} \leq \|\vec{a}\|\|\vec{b}\|$ apply Cauchy-Schwartz

$\boxed{\text{Independence and Dependence}}$

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M is invertible

Columns of M are

<u>independent</u>

no combination except

$0\vec{u}_1 + 0\vec{u}_2 + 0\vec{u}_3 = \vec{0}$

E is not invertible

Columns of E are

<u>dependent</u>

Other combinations give $\vec{0}$

for example

$2\vec{a}_1 + 0\vec{a}_2 + 0\vec{a}_3 = \vec{0}$

Three Equations in Three Unknowns

Example:

$$x + 2y + 3z = -3$$
$$3x + 2y + z = 3$$
$$2x + y + 3z = -3$$

Matrix equation $A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}}_{\substack{\text{coefficient} \\ \text{matrix} \\ A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}}_{\vec{b}}$$

The Row Picture

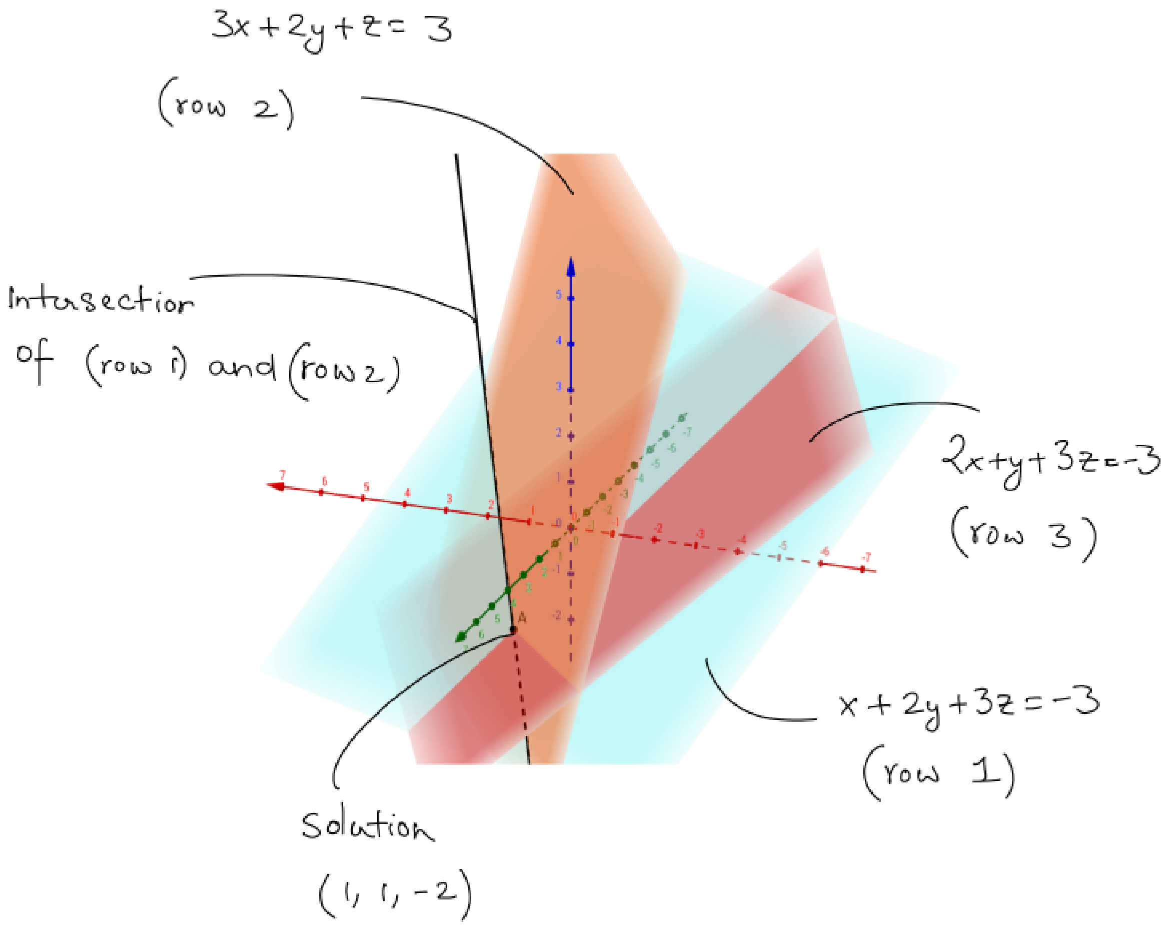
Consider the first row of the system of equations

$$x + 2y + 3z = -3,$$

or $\underbrace{(1, 2, 3)}_{\text{row 1}} \cdot \underbrace{(x, y, z)}_{\vec{x}} = -3$

this describes a <u>plane</u> in three-dimensional space

i.e., Inner products in $\mathbb{R}^3$ $\underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\text{Coefficients}} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{variables}} = d$ gives planes.



Column Picture

$$\boxed{\begin{aligned} x + 2y + 3z &= -3 \\ 3x + 2y + z &= 3 \\ 2x + y + 3z &= -3 \end{aligned}}$$

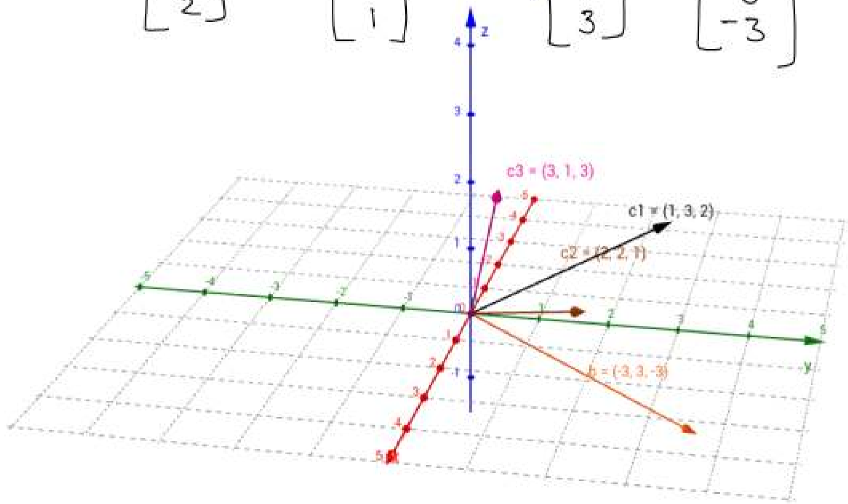
Vector form of the equations $A\vec{x} = \vec{b}$

$$x \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}$$

Which linear combination (i.e., choices of x, y, z) produce $\vec{b}$?

Correct combination is

$$1\begin{bmatrix}1\\3\\2\end{bmatrix} + 1\begin{bmatrix}2\\2\\1\end{bmatrix} + (-2)\begin{bmatrix}3\\1\\3\end{bmatrix} = \begin{bmatrix}-3\\3\\-3\end{bmatrix}$$



An Example where there is no solution...

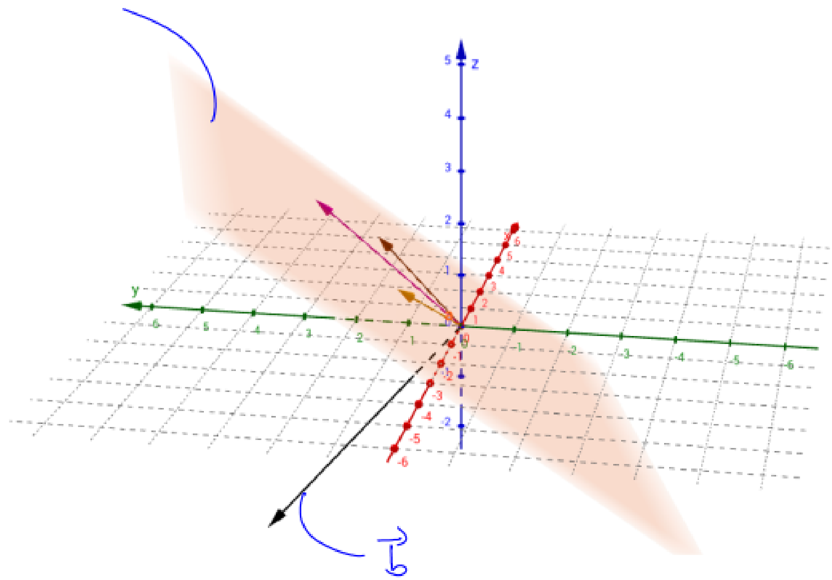
$$\underbrace{\begin{bmatrix}1 & 2 & -1\\3 & 2 & 1\\2 & 1 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}-3\\3\\-3\end{bmatrix}}_{\vec{b}}$$

col 3 = col 1 − col 2

all columns of the coefficient matrix A lie on a plane, but $\vec{b}$ does not lie on this plane

the column picture

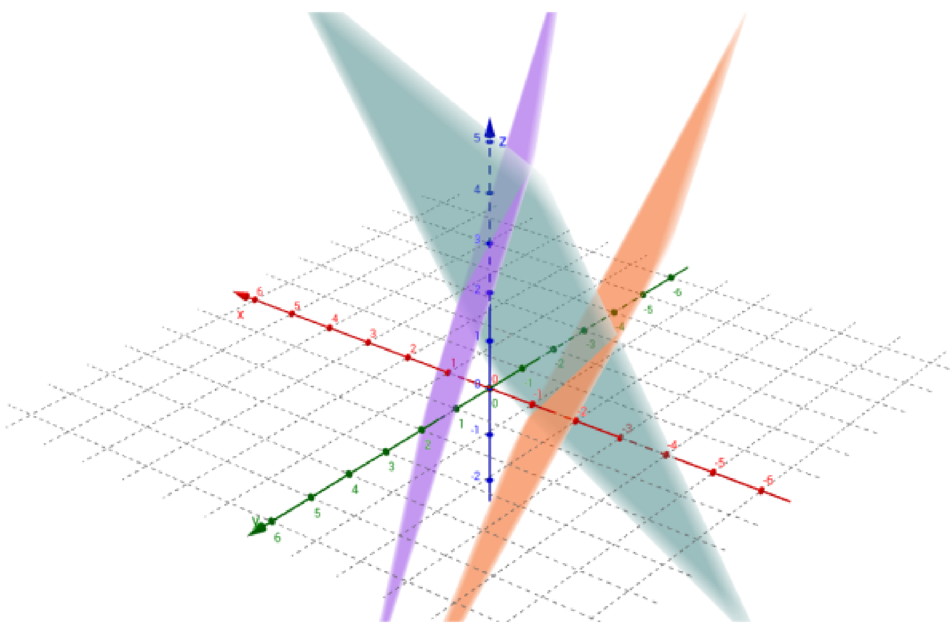
plane containing col 1, col 2, col 3



$\vec{b}$

(does not lie in plane)

the corresponding row picture



all the planes do not intersect at a single point

An example where there are infinitely many solutions.

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}_{\vec{b}}$$

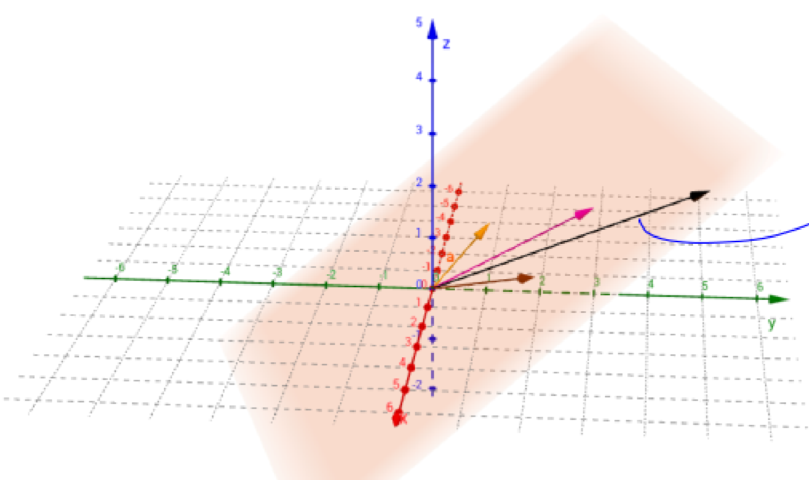
$\vec{b}$ is now a linear combination of only col 1 and col 2

col 3 = col 1 − col 2

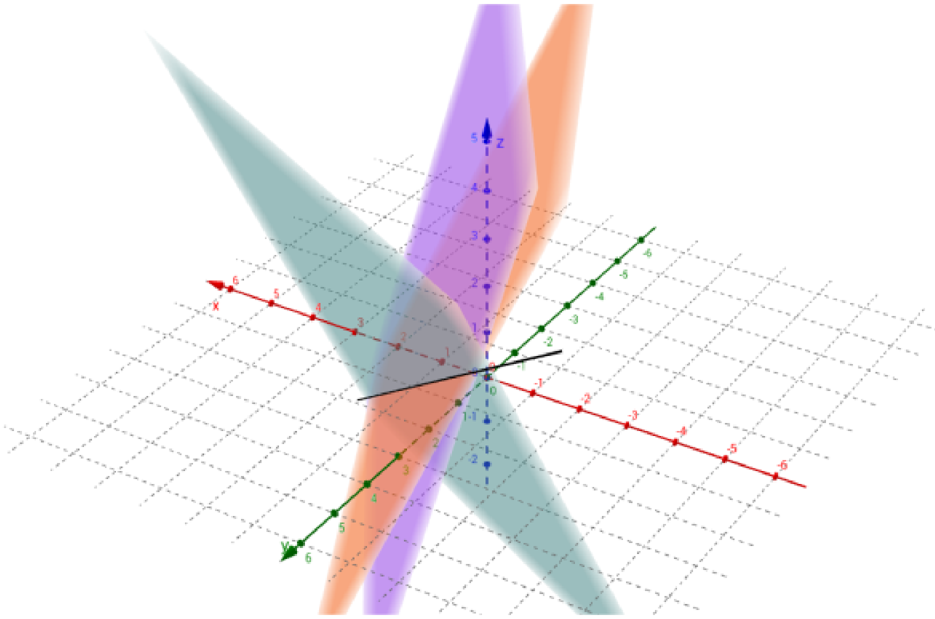
all columns of the coefficient matrix A lie on a plane, and $\vec{b}$ also lies on this plane

$$\underbrace{1\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}_{\text{Linear combination 1}} \qquad \underbrace{2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}_{\text{Linear combination 2}}$$

Infinitely many solutions!



$\vec{b}$ is now in the plane which contains col 1, col 2 and col 3 the corresponding row picture



all three planes intersect along a line!

$\boxed{\text{Review of Elimination}}$

Consider the system of equations

$$2x_1 + x_2 + 3x_3 = 1$$
$$4x_1 + 3x_2 + 5x_3 = 1$$
$$6x_1 + 5x_2 + 5x_3 = -3$$

With elimination, we reduce the given matrix equation to $\boxed{\text{upper triangular}}$ form, which can then be solved by $\boxed{\text{back substitution}}$

$$\begin{matrix} 2x_1 + x_2 + 3x_3 = 1 \\ 4x_1 + 3x_2 + 5x_3 = 1 \\ 6x_1 + 5x_2 + 5x_3 = -3 \end{matrix} \xrightarrow[\text{row 3} - \boxed{3} \times \text{row 1}]{\text{row 2} - \boxed{2} \times \text{row 1}} \begin{matrix} 2x_1 + x_2 + 3x_3 = 1 \\ x_2 - x_3 = -1 \\ 2x_2 - 4x_3 = -6 \end{matrix} \xrightarrow[\boxed{2} \times \text{row 2}]{\text{row 3} -} \begin{matrix} 2x_1 + x_2 + 3x_3 = 1 \\ 1x_2 - x_3 = -1 \\ -2x_3 = -4 \end{matrix}$$

multiplier

pivots

triangular

$\boxed{\text{Review of Elimination}}$

Consider the system of equations

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
4x_1 + 3x_2 + 5x_3 &= 1 \\
6x_1 + 5x_2 + 5x_3 &= -3
\end{aligned}
$$

With elimination, our reduce the given matrix aquation to $\boxed{\text{upper triangular}}$ form, which can then be solved by $\boxed{\text{back substitution}}$

Pivots

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
1x_2 - x_3 &= -1 \\
-2x_3 &= -4
\end{aligned}
\quad \xrightarrow[\text{Substitution}]{\text{Back}} \quad
\boxed{\begin{aligned} x_3 &= 2 \\ x_2 &= 1 \\ x_1 &= -3 \end{aligned}}
$$

$\boxed{\text{The Augmented Matrix}}$

$$
\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}
=
\underbrace{\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}}_{\vec{b}}
$$

$$
\boxed{\text{Augmented matrix} \quad [A \;\; \vec{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right]}
$$

- the augmented matrix has $\vec{b}$ as an extra column.

Let us perform elimination on the augmented matrix.

Elimination

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}
\quad \xrightarrow[\text{row ③} - \boxed{3} \times \text{row ①}]{\text{row ②} - \boxed{2} \times \text{row ①}} \quad
\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ x_2 - x_3 &= -1 \\ 2x_2 - 4x_3 &= -6 \end{aligned}
\quad \xrightarrow{\text{row ③} - \boxed{2} \times \text{row ②}} \quad
\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 1x_2 - x_3 &= -1 \\ -2x_3 &= -4 \end{aligned}$$

multiplier

pivots

triangular

Elimination using Matrices

$$E_{21}\left[A\ \vec{b}\right] = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 6 & 5 & 5 & -3 \end{array}\right]$$

$$E_{31}\left(E_{21}\left[A\ \vec{b}\right]\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 6 & 5 & 5 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -4 & -6 \end{array}\right]$$

$$E_{32}\left(E_{31}\,E_{21}\left[A\ \vec{b}\right]\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-2} & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -4 & -6 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

The Row Exchange Matrix

Recall that a row exchange was needed when elimination breaks down temporarily due to a 0 in the pivot

To exchange, for example, row ② and row ③, we use a permutation or row-exchange matrix of the form

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Here is the row-exchange matrix in action

... acting on a vector

$$P_{23}\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ \boxed{3} \\ \boxed{-4} \end{bmatrix}}_{\vec{b}} = \underbrace{\begin{bmatrix} 1 \\ \boxed{-4} \\ \boxed{3} \end{bmatrix}}_{\vec{b}_{new}}$$

... acting on an augmented matrix

$$P_{23}\left[A \ \ \vec{b}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & 10 & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 4 & 10 & 1 & 2 \end{array}\right]$$

In general,

The Row Exchange matrix $\boxed{P_{ij}}$ is the identity matrix with rows i and j reversed. When P_{ij} multiplies a matrix, it exchanges rows i and j.

Here is the row-exchange matrix which exchanges rows 2 and 4

$$P_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Recall:

Here is the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Interpretations of Matrix multiplication AB

① Dot product of rows of A with columns of B (previous slide)

② Matrix A times columns of B

$$A\underbrace{\left[\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_q\right]}_{B} = \underbrace{\left[A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_q\right]}_{AB}$$

③ Rows of A times matrix B

$$\left[\text{row } (i) \text{ of } A\right]\underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 9 & 7 & 5 \end{bmatrix}}_{B} = \left[\text{row } (i) \text{ of } AB\right]$$

LAWS OF MATRIX MULTIPLICATION

$AB \neq BA$ Commutative law not (usually) satisfied

$C(A+B) = CA + CB$

$(A+B)C = AC + BC$

Distributive laws

$A(BC) = (AB)C$ Associative law

Matrix Powers

$A^p = AAA\ldots A$ (p is a natural number)

$(A^p)(A^q) = A^{p+q}$

$(A^p)^q = A^{pq}$

BLOCK MATRICES

We can process matrices in "blocks".

For example, the augmented matrix $[\underbrace{A}_{\text{block 1}} \ \underbrace{\vec{b}}_{\text{block 2}}]$

Note: blocks 1 and 2 have different sizes

block of matrix

$$\text{Block Multiplication} \quad \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} B_{11} & B_{12} \cdots \\ B_{21} & B_{22} \cdots \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \cdots \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \cdots \end{bmatrix}}_{AB}$$

Notes on Inverses

Note 1 The inverse exists if and only if elimination produces n pivots

Note 2 If A is invertible, the one and only solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

Why? $A\vec{x} = \vec{b} \xrightarrow[\text{by } A^{-1}]{\text{multiply}} A^{-1}(A\vec{x}) = A^{-1}\vec{b}$ or $(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

Note 3 Suppose there is a non-zero vector $\vec{x}$ such that $A\vec{x} = \vec{0}$.

Then A cannot have an inverse.

Note 4 If A is diagonal, $A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_n \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & 1/d_2 & \\ & & \ddots \\ 0 & & & 1/d_n \end{bmatrix}$

Note 5 If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$

Why? $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$

Faster Method When making the augmented matrix upper triangular, our work only depends on A.

Therefore, we can solve for all the equations at once!

Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ using elimination

$\boxed{\text{Step 1}}$ Construct augmented matrix

$[A \; I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$

$\boxed{\text{Step 2}}$ Perform elimination

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

Gaussian elimination

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \quad \text{row ② - 2 row ①}$$

finish with back substitution

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right] \quad \text{row ③ + row ②}$$

Step 3 Perform back substitution bottom to top; want zeros above pivots too

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -4 & 2 & 1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right] \quad \text{row ② + row ③}$$

Gauss – Jordan Method

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & -2 & 0 & -4 & 2 & 1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right] \quad \text{row ① + row ②}$$

Reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\boxed{\begin{matrix} -3 & 2 & 1 \\ 2 & -1 & -\frac{1}{2} \\ 2 & -1 & -1 \end{matrix}}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right] \quad \text{Divide each row by pivot}$$

Review - Elimination

Let's perform elimination / row reduction on $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21} A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 2 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \qquad E_{31} E_{21} A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \qquad E_{32} E_{31} E_{21} A = \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U}$$

pivots

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

upper triangular form

With no row exchanges,

$$(E_{32} E_{31} E_{21}) A = U$$

Move over the E's to other side

$$A = (E_{32} E_{31} E_{21})^{-1} U$$

$$= \underbrace{(E_{21}^{-1} E_{31}^{-1} E_{32}^{-1})}_{L} U$$

$$= L U$$

We know that

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

lower triangular

$$E_{21}^{-1} E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{\frac{1}{2}} & 1 & 0 \\ \boxed{\frac{1}{2}} & \boxed{\frac{1}{3}} & 1 \end{bmatrix}$$

$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$ is lower triangular

* diagonal contains 1's
* multipliers unchanged

$\boxed{\text{LU Factorization}}$

$$\boxed{A = LU}$$

Elimination without row exchanges

Upper triangular U has pivots on its diagonal

Lower triangular L has 1's on its diagonal

The multipliers ℓ_{ij} are below the diagonal of L

$\boxed{\text{LDU Factorization}}$

Consider the coefficient matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -5/3 & 1 \end{bmatrix} \qquad E_{21}A = \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix}}_{U}$$

$$\Rightarrow A = E_{21}^{-1} \begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 5/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix}}_{U}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 5/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}}_{\tilde{U}}$$

Split U into $D\tilde{U}$

$$\begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}$$

pivots

$\boxed{\text{TRANSPOSE}}$

$\boxed{\text{Def}^n}$ Let $A \in \mathbb{R}^{m \times n}$. The <u>transpose</u> of A is $A^T \in \mathbb{R}^{n \times m}$ formed by exchanging rows with columns.

$$\boxed{(A^T)_{ij} = A_{ji}}$$

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 0 \\ -1 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 4 \end{bmatrix}$$

Column 1

entry 3, 2

row 1

entry 2, 3

Properties of the Transpose of Matrices

$\boxed{\text{Sum}}$ $\quad (A+B)^T = A^T + B^T$

$\boxed{\text{Product}}$ $\quad (AB)^T = B^T A^T$ (also extends to 3 or more factors for example: $A = LDU$; $A^T = U^T D^T L^T$)

$\boxed{\text{Inverse}}$ $\quad (A^{-1})^T = (A^T)^{-1}$

Understanding the product rule

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & -4 \end{bmatrix}$$

col 1

$$0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -1 & -4 \end{bmatrix}$$

row 1

$$0\begin{bmatrix} 1 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \end{bmatrix}$$

Inner and Outer Products

(Recall) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. We denoted the inner product by $\vec{x} \cdot \vec{y}$

Using transpose notation, we have

dot product or inner product is $\vec{x}^T \vec{y}$

$\vec{x}^T \in \mathbb{R}^{1 \times n}$ $\qquad \vec{y} \in \mathbb{R}^{n \times 1}$

$\vec{x}^T \vec{y} \in \mathbb{R}^{1 \times 1}$

Note: The outer product is $\vec{x}\,\vec{y}^T$

$\vec{x} \in \mathbb{R}^{n \times 1}$ $\qquad \vec{y}^T \in \mathbb{R}^{1 \times n}$

$\vec{x}\,\vec{y}^T \in \mathbb{R}^{n \times n}$

SYMMETRIC MATRICES

Def^n A matrix $S \in \mathbb{R}^{n \times n}$ is called symmetric if $S^T = S$

For example, consider the row exchange matrix

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{32}^T$$

Recall also that $P_{32}^{-1} = P_{32}$

A row exchange or swap matrix is always symmetric

<u>Another example</u> Consider any $A \in \mathbb{R}^{m \times n}$. Then

transpose of $\underbrace{A^T A}_{\in \mathbb{R}^{n \times n}}$ is $(A^T A)^T = A^T (A^T)^T = A^T A$

Transpose of $\underbrace{A A^T}_{\in \mathbb{R}^{m \times m}}$ is $(A A^T)^T = A A^T$

PERMUTATION MATRICES

A permutation matrix is the identity matrix with its rows reordered any way you want.

A row exchange or swap matrix is a special case

Examples: (2x2 matrices)

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Identity matrix}} \qquad \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{swap matrix}}$$

2 total permutations

PERMUTATION MATRICES

A permutation matrix is the identity matrix with its rows reordered any way you want.

A row exchange or swap matrix is a special case

Examples: (3×3 matrices)

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ swap $P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ swap

$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ swap

$P_{32} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ not swap, not symmetric, but product of swaps

$P_{21} P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

6 total permutations

$6 = 3!$
$= (3)(2)(1)$

(row order) 123 132 213 231 312 321

Thm Every permutation matrix P has $P^{-1} = P^T$

Proof Let P be a permutation matrix.

Then $P = S_1 \cdots\cdots S_{n-1}$ ($n-1$ swaps

Hence, $P^{-1} = (S_1 S_2 \cdots\cdots S_{n-1})^{-1}$

$= S_{n-1}^{-1} \cdots\cdots S_2^{-1} S_1^{-1}$ $\quad \left((AB)^{-1} = B^{-1} A^{-1} \right)$

$= S_{n-1} \cdots\cdots S_2 S_1$ $\quad$ (each swap matrix is its own inverse)

$= S_{n-1}^T \cdots\cdots S_2^T S_1^T$ $\quad$ (each swap matrix is its own transpose)

$= (S_1 S_2 \cdots\cdots S_{n-1})^T$ $\quad \left((AB)^T = B^T A^T \right)$

$= P^T$

PA = LU FACTORIZATION

Every matrix $A \in \mathbb{R}^{n \times n}$ has a $\underbrace{P}_{\text{permutation matrix}}\underbrace{L}_{\text{lower triangular}}\underbrace{D}_{\text{diagonal}}\overbrace{U}^{\text{upper triangular}}$ factorization

Example $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

Swap rows ② and ③ $\quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_{32}A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

row ② - 2 row ① $\quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}P_{32}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

Scale with diagonal pivots $\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad E_{21}P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

PA = LU FACTORIZATION

Every matrix $A \in \mathbb{R}^{n \times n}$ has a $\underbrace{P}_{\text{permutation matrix}}\underbrace{L}_{\text{lower triangular}}\underbrace{D}_{\text{diagonal}}\overbrace{U}^{\text{upper triangular}}$ factorization

Example $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

$$P_{32}A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Using $P_{32}^{-1} = P_{32}^{T} = P_{32}$, we have $\quad A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$

Vector Space Axioms

Let V be a set on which the operations of addition and scalar multiplication are defined. By this we mean that, with each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, we can associate a unique element $\mathbf{x}+\mathbf{y}$ that is also in V, and with each element $\mathbf{x}$ in V and each scalar α, we can associate a unique element $\alpha\mathbf{x}$ in V. The set V, together with the operations of addition and scalar multiplication, is said to form a **vector space** if the following axioms are satisfied:

A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V.
A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in V.
A3. There exists an element $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x}\in V$.
A4. For each $\mathbf{x}\in V$, there exists an element $-\mathbf{x}$ in V such that $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.
A5. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}$ for each scalar α and any $\mathbf{x}$ and $\mathbf{y}$ in V.
A6. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x}\in V$.
A7. $(\alpha\beta)\mathbf{x}=\alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x}\in V$.
A8. $1\cdot\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

If S is a nonempty subset of a vector space V, and S satisfies the conditions

(i) $\alpha\mathbf{x}\in S$ whenever $\mathbf{x}\in S$ for any scalar α

(ii) $\mathbf{x}+\mathbf{y}\in S$ whenever $\mathbf{x}\in S$ and $\mathbf{y}\in S$

then S is said to be a **subspace** of V.

$\boxed{\text{Def}^n}$ Given two vectors $\vec{u},\vec{v}$ we define

$$\text{span}(\vec{u},\vec{v}) = \text{the set of all possible linear combinations of } \vec{u},\vec{v}$$
$$= \{ a\vec{u}+b\vec{v} \text{ for any scalars } a,b \}$$

The span of $\vec{u}_1,\vec{u}_2,\ldots\ldots,\vec{u}_n$ is the set of all linear combinations of $\vec{u}_1,\vec{u}_2,\ldots\ldots,\vec{u}_n$

<u>Note:</u> If V is a vector space, then $\text{span}(\vec{u},\vec{v}) \subseteq V$ for all $\vec{u},\vec{v}\in V$

A Subspace containing $\vec{u},\vec{v}$ must contain $\text{span}(\vec{u},\vec{v})$

COLUMN SPACE OF A

The column space of A contains all linear combinations of the columns of A

The column space of A is denoted as $C(A)$

Let $A \in \mathbb{R}^{m \times n}$. Using the function view of a matrix

$$\vec{x} \in \mathbb{R}^n \longrightarrow \boxed{A \in \mathbb{R}^{m \times n}} \longrightarrow \vec{y} \in \mathbb{R}^m \qquad \vec{y} = A\vec{x}$$

$$A\vec{x} = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots\cdots & \vec{a}_n \\ | & | & & | \end{pmatrix} \vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots\cdots + x_n \vec{a}_n$$

$\vec{a}_i \in \mathbb{R}^m$

$$= \text{span}(\vec{a}_1, \ldots\ldots, \vec{a}_n)$$

This is a subspace of $\mathbb{R}^m$

Alternate notation <u>Range</u> of A $\qquad A\vec{x}$ always in $C(A)$

<u>Why are we interested in $C(A)$?</u>

The system $A\vec{x} = \vec{b}$ is solvable if and only if $\vec{b}$ is in the column space of A.

<u>Example:</u> $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$C(A) = \text{span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$$

= plane containing the the columns $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

C(A)