REVIEW PROBLEMS

CHAPTER 1

① Find unit vectors $\vec{u}_1$ and $\vec{u}_2$ in the directions of $\vec{v} = (2, 1)$ and $\vec{w} = (1, 3, -1)$. Find unit vectors $\vec{c}_1$ and $\vec{c}_2$ that are perpendicular to $\vec{u}_1$ and $\vec{u}_2$.

$\boxed{\vec{v} = (2, 1)}$ $\quad \|\vec{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$. Hence $\vec{u}_1 = \dfrac{\vec{v}}{\|\vec{v}\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

Choose $\vec{c}_1 = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$. Then $\vec{c}_1 \cdot \vec{u}_1 = 0$ and $\|\vec{c}_1\| = 1$.

$\boxed{\vec{w} = (1, 3, -1)}$ $\quad \|\vec{w}\| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}$. Hence $\vec{u}_2 = \dfrac{\vec{w}}{\|\vec{w}\|} = \left(\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right)$.

Note that $(1, 0, 1)$ is perpendicular to $\vec{w}$. Hence $\vec{c}_2 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ is perpendicular to $\vec{u}_2$.

(2) All possible linear combinations of $\vec{a} = (1,1,1,1)$, $\vec{b} = (1,1,2,2)$, $\vec{c} = (0,0,1,1)$ and $\vec{d} = (3,3,3,3)$ lie on a

LINE $\qquad$ $\boxed{\text{PLANE}}$ $\qquad$ 3D SPACE $\qquad$ 4D SPACE

(3) Suppose that $\|\vec{a}\| = 3$ and $\|\vec{b}\| = 4$. The maximum possible value of $\|\vec{a}+\vec{b}\| + \vec{a}\cdot\vec{b}$ is ______

By the triangle inequality, $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Schwarz inequality, $|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$

Equality occurs when $\vec{a}$ and $\vec{b}$ point in the same direction

then $\|\vec{a}+\vec{b}\| + \vec{a}\cdot\vec{b} = (3+4) + (3 . 4 . \cos 0^\circ)$

$= 7 + 12 = 19$

① Which number forces a row exchange, and what is the triangular system (non-singular) for that d? Which d makes this system singular (no third pivot)?

CHAPTER 2

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

Let's perform elimination on the coefficient matrix $A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & d & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & d & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{row ② - 2 row①}} \begin{bmatrix} 2 & 5 & 1 \\ 0 & d-10 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

If $d=10$, we need a row exchange to continue elimination (rows ② and ③)

If $d=11$, then we have $\begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. The last two rows are dependent; the system is singular.

② Apply row elimination to reduce this matrix from A to I. Then write $A^{-1}$ as a product of three (or more) elimination matrices coming from that elimination. Multiply these matrices to find $A^{-1}$.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Swap rows ① and ② $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

row ③ → −2 row ① + row ③ $\quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_{31}PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

row ② → −row ③ + row ② $\quad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{23}E_{31}PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Hence, $A^{-1} = E_{23}E_{31}P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$

③ Write down the inverse of $A \in \mathbb{R}^{3\times 3}$ <u>without</u> performing elimination. Justify your answer.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

A is a permutation matrix (its rows are the rows of the identity matrix, but re-ordered).

Hence $$A^{-1} = A^{T} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

④ Short answer / answer only (fill in the blanks, true/false) problems

(a) Let $A, B, C \in \mathbb{R}^{m\times n}$. Write $(AB+C)^T$ in terms of $A^T, B^T, C^T$

$$(AB+C)^T = (AB)^T + C^T$$
$$= B^T A^T + C^T$$

(b) True or $\boxed{\text{False}}$ If AB and BA are defined, then A and B are square

If $\underbrace{A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}}_{\text{non-square}}$ then $\underbrace{AB \in \mathbb{R}^{m\times m}, BA \in \mathbb{R}^{n\times n}}_{\text{square}}$

(c) True or $\boxed{\text{False}}$ If $AB = B$, then $A = I$

Counterexample: $B = 0$ matrix

① Let $\mathbb{Z}$ denote the set of all integers with the usual definition of addition, and define scalar multiplication, denoted $\circ$, by

$$c \circ k = \lfloor c \rfloor \cdot k \quad \text{for all } k \in \mathbb{Z},$$

where $\lfloor c \rfloor$ denotes the greatest integer less than or equal to $c$.

For example, $1.25 \circ 7 = \lfloor 1.25 \rfloor \circ 7 = 1 \cdot 7 = 7$.

Is $\mathbb{Z}$, together with these operations, a vector space? Justify your answer.

CHAPTER 3

No. $\mathbb{Z}$ with these operations is not a vector space.

Here is an axiom that fails

$$(c+d) \circ x = c \circ x + d \circ x \quad \text{for } x \in \mathbb{Z}$$

Choose $c = 1.25$, $d = 0.75$. Then $(c+d) \circ x = (1.25 + 0.75) \circ x = 2 \circ x = \lfloor 2 \rfloor x = 2x$

However, $c \circ x + d \circ x = \lfloor 1.25 \rfloor \cdot x + \lfloor 0.75 \rfloor \cdot x = 1 \cdot x + 0 \cdot x = x$

② Determine whether the following sets are subspaces of $\mathbb{R}^2$.

(a) $\{(x_1, x_2) : x_1 + x_2 = 0\} = V$

$\boxed{\text{Yes}}$ Let $\vec{a}, \vec{b} \in V$ $\vec{a} = (x_1, x_2)$, $\vec{b} = (y_1, y_2)$

Then $\vec{a} + \vec{b} = (x_1 + y_1, x_2 + y_2)$ with $(x_1+y_1) + (x_2+y_2) = (x_1+x_2) + (y_1+y_2) = 0$

Hence $\vec{a} + \vec{b} \in V$

Similarly,

let $c \in \mathbb{R}$. $c\vec{a} = (cx_1, cx_2)$, with $cx_1 + cx_2 = c(x_1 + x_2) = c\,0 = 0$

Hence $c\vec{a} \in V$

(b) $\{(x_1, x_2) : x_1^2 = x_2^2\} = A$

$\boxed{\text{No}}$ Let $\vec{p}, \vec{q} \in A$ where $\vec{p} = (2, 2)$ and $\vec{q} = (3, -3)$

$\vec{p} \in A$ since $2^2 = 2^2$ $\quad \vec{q} \in A$ since $3^2 = (-3)^2$

$\vec{p} + \vec{q} = (2, 2) + (3, -3) = (5, -1)$

$\vec{p} + \vec{q} \notin A$ since $5^2 \neq (-1)^2$. No closure under addition

③ Column Space of A

Describe the column space of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Is the vector $(2, 3, 1)$ in the column space of this matrix?

$C(A) = \text{span}\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$

Only vectors $(b_1, b_2, b_3)$ with $b_3 = 0$ are in the column space of this matrix.

Hence $(2, 3, 1) \notin C(A)$.