NAME: SOLUTIONS Quiz 9, MTH314-001

1. Let

$$A = \begin{pmatrix} 1 & -3 & -5 \\ 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} -6 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

If the Gram-Schmidt process is applied to determine an orthonormal basis for $C(A)$ and a QR factorization of A, then, after the first two orthonormal vectors $\vec{q}_1$ and $\vec{q}_2$ are computed, we have

$$Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \underline{-1/2} \\ \frac{1}{2} & \frac{1}{2} & \underline{-1/2} \\ \frac{1}{2} & -\frac{1}{2} & \underline{1/2} \\ \frac{1}{2} & \frac{1}{2} & \underline{1/2} \end{pmatrix}, \qquad R = \begin{pmatrix} 2 & -2 & \underline{-1} \\ 0 & 4 & \underline{3} \\ 0 & 0 & \underline{6} \end{pmatrix}.$$

(a) Finish the process. Determine $\vec{q}_3$ and fill in the third columns of Q and R. *[6 points]*

Let $\vec{a}_3 = \begin{pmatrix} -5 \\ -2 \\ 1 \\ 4 \end{pmatrix}$, $\vec{q}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$ and $\vec{q}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$

Then $\vec{q}_3 = \dfrac{\vec{a}_3 - P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3}{\left\| \vec{a}_3 - P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3 \right\|}$ where $P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3 = \vec{p}_2 = r_{13}\vec{q}_1 + r_{23}\vec{q}_2$

$$r_{13} = \vec{a}_3 \cdot \vec{q}_1 = \vec{a}_3^T \vec{q}_1 = -5/2 - 2/2 + 1/2 + 4/2 = -1$$

$$r_{23} = \vec{a}_3 \cdot \vec{q}_2 = \vec{a}_3^T \vec{q}_2 = 5/2 - 2/2 - 1/2 + 4/2 = 3$$

$$P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3 = (-1)\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + 3\begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{a}_3 - P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3 = \begin{pmatrix} -5 \\ -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \\ 3 \end{pmatrix}$$

$$r_{33} = \left\| \vec{a}_3 - P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3 \right\| = \sqrt{(-3)^2 + (-3)^2 + 3^2 + 3^2} = \sqrt{4(9)} = 6$$

Hence, $\vec{q}_3 = \dfrac{\vec{a}_3 - P_{\mathrm{span}\{\vec{q}_1, \vec{q}_2\}} \vec{a}_3}{r_{33}} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$

(b) Use the QR factorization to find the least squares solution of $A\vec{x} = \vec{b}$. *[4 points]*

Least-squares solution is $A^T A \vec{x} = A^T \vec{b}$

Using $A = QR$, we get

$$(QR)^T (QR)\vec{x} = (QR)^T \vec{b}$$

or, $R^T (Q^T Q) R \vec{x} = R^T Q^T \vec{b}$

or, $R^T R \vec{x} = R^T Q^T \vec{b}$ (since Q is orthogonal)

or, $R\vec{x} = Q^T \vec{b}$

$$\begin{bmatrix} 2 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -6 \\ 1 \\ 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}$$

Using back substitution, $x_3 = 1$

$$4x_2 = 6 - 3x_3$$

$$\Rightarrow 4x_2 = 6 - 3$$

$$\Rightarrow x_2 = 3/4$$

and $2x_1 - 2x_2 - x_3 = 1$

$$\Rightarrow 2x_1 = 1 + 2\left(3/4\right) + 1$$

$$\Rightarrow 2x_1 = \frac{7}{2}$$

$$\Rightarrow x_1 = \frac{7}{4}$$

Least squares sol^n.

$$\vec{x} = \begin{bmatrix} 7/4 \\ 3/4 \\ 1 \end{bmatrix}$$