NAME: SOLUTIONS Quiz 8, MTH314-001

1. Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

(a) Project $\vec{b}$ onto the $C(A)$ by solving $A^T A\vec{x} = A^T\vec{b}$ and $\vec{p} = A\vec{x}$. Show all steps. *[5 points]*

We have $A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 45 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 42 \end{bmatrix}$

$A^TA\hat{x} = A^T\vec{b} \iff \begin{bmatrix} 3 & 9 \\ 9 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 42 \end{bmatrix}$

(Solve by elimination) $\left[\begin{array}{cc|c} 3 & 9 & 10 \\ 9 & 45 & 42 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|c} 3 & 9 & 10 \\ 0 & 18 & 12 \end{array}\right]$

by back substitution, $x_2 = 2/3$ and $x_1 = \dfrac{10 - 9(2/3)}{3} = 4/3$

$\vec{x} = \begin{bmatrix} 4/3 \\ 2/3 \end{bmatrix}$

$\vec{p} = A\vec{x} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 4/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 10/3 \\ 16/3 \end{bmatrix}$

(b) Write down an analytical expression for the projection matrix P. (Note: you do not have to numerically evaluate/solve for the matrix.) *[1 point]*

$$P = A(A^TA)^{-1}A^T$$

Note: in part (a), we solved

$$(A^TA)\vec{x} = A^T\vec{b}$$
$$\Rightarrow \vec{x} = (A^TA)^{-1}A^T\vec{b}$$

and then evaluated

$$\vec{p} = A\vec{x} = \underbrace{A(A^TA)^{-1}A^T}_{P}\vec{b}$$

(c) Find the error vector $\vec{e} = \vec{b} - \vec{p}$ and compute its dot product with the columns of A. What does this tell you about the relationship between $\vec{e}$ and the columns of A? *[4 points]*.

$$\vec{e} = \vec{b} - \vec{p}$$
$$= \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix}$$
$$\vec{e} = \begin{pmatrix} -1/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

Note:

$$\vec{e} \cdot \vec{a_1} = (-1/3, 2/3, -1/3) \cdot (1, 1, 1)$$
$$= -1/3 + 2/3 - 1/3 = 0$$
$$\vec{e} \cdot \vec{a_2} = (-1/3, 2/3, -1/3) \cdot (0, 3, 6)$$
$$= 2 - 2 = 0$$

$$\vec{e} \cdot \vec{a_1} = 0$$
$$\vec{e} \cdot \vec{a_2} = 0$$

Since the error is perpendicular to the column space of A (subspace spanned by $\vec{a_1}, \vec{a_2}$)