NAME: SOLUTIONS   Quiz 7, MTH314-001

1. Answer the questions about $A \in \mathbb{R}^{3\times 3}$ below. *[10 points]*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Performing elimination, $[A \vdots \vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{array}\right]$

$$\xrightarrow[R_3 = R_3 - 7R_1]{R_2 = R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{array}\right]$$

$$\xrightarrow[R_3 = R_3 - 2R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 2b_2 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 / -3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & -\frac{b_2}{3} + \frac{4}{3}b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 2b_2 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{5}{3}b_1 + \frac{2}{3}b_2 \\ 0 & 1 & 2 & -\frac{b_2}{3} + \frac{4}{3}b_1 \\ 0 & 0 & 0 & b_3 + b_1 - 2b_2 \end{array}\right] \begin{array}{l} \leftarrow \text{pivot} \\ \leftarrow \text{rows} \\ \end{array}$$

pivot cols.

(a) Write a basis for $C(A)$. *[2 points]*

$$\text{Basis for } C(A) = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$$

pivot cols. are cols 1 and 2 we therefore look at the corresponding columns of A

(b) Write a basis for $C(A^{\mathrm{T}})$. *[2 points]*

$$\text{Basis for } C(A^T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

we look for the pivot rows in the reduced row echelon form

(c) Write a basis for $N(A)$. *[2 points]*

We have 1 free column and hence one special solution

Special solution $\vec{s}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Hence, Basis for $N(A) = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

We look for solutions to $A\vec{x} = \vec{0}$

pivot vars. x_1 and x_2

free vars. x_3

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}\right\} \xrightarrow{\text{Set } x_3 = 1} \begin{aligned} x_1 &= 1 \\ x_2 &= -2 \end{aligned}$$

(d) Write a basis for $N(A^{\mathrm{T}})$. *[2 points]*

From the last row of the reduced row echelon form,

we have (row 1) $-$ 2 (row 2) + (row 3) = zero row

Hence basis for $N(A^T) = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

Check that $A^T \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \vec{0}$

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(e) Fill in the blanks *[0.5 points each]*

i. The dimension of $C(A)$ is ___2___ (there are vectors in basis for $C(A)$)

ii. The dimension of $N(A^{\mathrm{T}})$ is ___1___ (there is 1 vector in basis for $N(A^T)$)

iii. $N(A)^{\perp} = \{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} \perp N(A)\} = C(A^T) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

iv. $N(A^{\mathrm{T}})^{\perp} = \{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} \perp N(A^{\mathrm{T}})\} = C(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$