NAME: SOLUTIONS — Quiz 6, MTH314-001

1. Find every possible solution, $\vec{x}$, to the system below. *[6 points]*

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 8 & 5 & 4 \\ 0 & 0 & -1 & -2 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Performing elimination on the augmented matrix $[A \vdots \vec{b}]$, we get

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 1 \\ 2 & 8 & 5 & 4 & 1 \\ 0 & 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 4 & 0 & -3 & 3 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We have reached reduced row echelon form.

- Pivot columns: col 1 and col 3
- Free columns (variables): col 2 and col 4

Particular solution $\vec{x}_p$

$\vec{x}_p$ is obtained by setting $x_2 = x_4 = 0$.

We get $\vec{x}_p = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Nullspace solutions $\vec{x}_n$

$\vec{x}_n = x_2 \vec{s}_1 + x_4 \vec{s}_2$ where $x_2, x_4 \in \mathbb{R}$ and $\vec{s}_1, \vec{s}_2$ are special solutions.

setting $x_2 = 1, x_4 = 0$, we get $\vec{s}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = 0, x_4 = 1$, we get $\vec{s}_2 = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Hence $\vec{x}_n = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

The complete solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$= \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2. Answer the questions about $A \in \mathbb{R}^{2\times 4}$ below. *[2 points each]*

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \end{pmatrix}$$

Performing elimination,

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & -2 & 4 \end{bmatrix} \xrightarrow[R_2 \to R_2/2]{} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

We have reached reduced row echelon form.

(a) Write a basis for $C(A)$.

- Pivot columns of col 1 and col 2.

Hence basis for $C(A)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$

(b) Write a basis for $N(A)$.

- Free variables are $x_3$ and $x_4$

Special solution $\vec{s}_1 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (obtained by setting $x_3 = 1$, $x_4 = 0$)

$\vec{s}_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ (obtained by setting $x_3 = 0$, $x_4 = 1$)

Basis for $N(A)$ is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$