NAME: SOLUTIONS **Quiz 5**, MTH314-001

1. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}.$$

(a) Use elimination to find the reduced row echelon matrix R. Identify the pivot columns and free columns. *[3 points]*

(b) How many special solutions does $A\vec{x} = \vec{0}$ have? Write the nullspace $N(A)$ as the span of the special solution(s) *[2 points]*

(c) Write the column space, $C(A)$, of A as the span of two vectors. *[1 point]*

(a) Performing elimination, we have

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_2]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to R_2/(-1)]{} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Columns 1 and 3 are the pivot columns.
Colums 2 and 4 are the free columns.

(b) Since there are two free columns, there are 2 special solutions.

we have $\vec{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{s}_2 = \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Hence $N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

(c) $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$

Note that these are the first and third columns of A (columns 1 and 3 are the pivot columns in R)

2. Suppose an m by n matrix has r pivots. Answer the following questions: *[1 point each]*
   (a) The number of special solutions is $n-r$. ($n-r$ free columns)
   (b) The nullspace contains only $\vec{x} = \vec{0}$ when $r =$ n. (no free columns)
   (c) The column space is all of $\mathbb{R}^m$ when $r =$ m. (there are no zero rows or all rows are pivot rows)

3. Construct a matrix A such that its nullspace contains all multiples of $(2, -1, 3, 1)$. *[1 point]*.

Here is such a matrix: $A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

<u>Note</u>: A has 3 pivot columns and 1 free column. Hence $N(A)$ contains all multiples of the special solution $\vec{s}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$