1. Compute $A^{-1} \in \mathbb{R}^{3\times 3}$ using elimination. Show your work (write down the row operations used in each step). *[4 points]*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will use Gauss-Jordan elimination on the augmented matrix

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad (-\text{row } ① + \text{row } ②)$$

Note that A has been reduced to upper triangular form. We now proceed with elimination to obtain zeros above the pivots.

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad (-\text{row } ③ + \text{row } ②)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -2 \\ 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad (2\text{ row } ② + \text{row } ①)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad (\text{divide by } -1)$$

$$= [I \,\vdots\, A^{-1}]$$

Hence $A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

SANITY CHECK:

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

2. Answer the following True/False questions about $M, N, P \in \mathbb{R}^{n\times n}$ by circling the correct answer. **When the answer is false, mention why!** *[1 point each]*

(a) It is always true that $(MP)^2 = M^2P^2$. T F

$$(MP)^2 = (MP)(MP) = M\underset{\sim}{PM}P \neq M\underset{\sim}{M}PP = M^2P^2$$

since $PM \neq MP$ in general

for example:

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad MP = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M^2P^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (MP)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Consider the block matrices $(M \mid N) \in \mathbb{R}^{n\times 2n}$ and $\begin{pmatrix} M \\ P \end{pmatrix} \in \mathbb{R}^{2n\times n}$.

Then, $(M \mid N)\begin{pmatrix} M \\ P \end{pmatrix} = M^2 + NP$. T F

This follows from the definition of block matrices and block multiplication.

(c) $(MP)^{-1}$ can still exist if row 2 of M is three times row 1 of M. T F

Interpreting matrix multiplication as a row times matrix

[row ⓘ of M] P = [row ⓘ of MP].

Then row ② of MP is three times row ① of MP.

This means that MP does not have a full set of pivots, and therefore not invertible.

Ex: $M = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad MP = \begin{bmatrix} 7 & 10 \\ 21 & 30 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 7 & 10 \\ 0 & 0 \end{bmatrix}$

3. Use elimination to compute the LDU factorization of $A \in \mathbb{R}^{2\times 2}$ below. That is, find a lower triangular matrix $L \in \mathbb{R}^{2\times 2}$, a diagonal matrix $D \in \mathbb{R}^{2\times 2}$, and an upper triangular matrix $U \in \mathbb{R}^{2\times 2}$ with ones on the diagonal, such that $A = LDU$. Show all your steps. *[3 points]*

$$A = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}.$$

Let's perform elimination on A.

Using $E_{21} = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix}$ we have

$$E_{21} A = \begin{bmatrix} 3 & 1 \\ 0 & -5/3 \end{bmatrix}$$

Using $E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix}$, we get

$$A = E_{21}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & -5/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -5/3 \end{bmatrix}$$

this is now a LU factorization of A.

Since $\begin{bmatrix} 3 & 1 \\ 0 & -\frac{5}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -\frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$, we have

$$A = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -\frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix} = LDU.$$