NAME: $\boxed{\text{SOLUTIONS}}$ **Quiz 13**, MTH314-001

1. Answer the following questions about $A \in \mathbb{R}^{3\times 3}$ below.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

(a) Is A positive definite? Justify your answer. *[2 points]*

$\boxed{\text{No}}$ A is a symmetric matrix with a negative pivot in row 2. Hence at least one of its eigenvalues are negative. Therefore A is not positive definite.

(Alternatively) find eigenvalues

$$|A-\lambda I| = 0 \Rightarrow (1-\lambda)[(-1-\lambda)(-\lambda)-4] + 2(2)(1+\lambda) = 0$$

$$\Rightarrow \lambda(1+\lambda)(1-\lambda) - 4(1-\lambda) + 4(1+\lambda) = 0$$

$$\Rightarrow \lambda(1-\lambda^2+8) = 0$$

$$\Rightarrow \lambda(9-\lambda^2) = 0, \text{ or } \lambda = 0 \text{ or } \lambda = 3 \text{ or } \lambda = -3$$

Since there is a non-positive eigenvalue, A is not positive definite.

(b) Calculate the sum of the eigenvalues of A. Show your work! *[2 points]*

Sum of eigenvalues, $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A)$

$= 1 - 1 + 0$

$= 0$

Note: trace (A) = Sum of diagonal elements of A

(c) Calculate the product of the eigenvalues of A. Show your work! *[3 points]*

$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$

Note: $|A| = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{vmatrix} = -4 + 2(2) = 0$

product of eigenvalues, $\lambda_1 \lambda_2 \lambda_3 = |A| = 0$

(d) The matrix A can be diagonalized as $A = Q\Lambda Q^{\mathrm{T}}$. Fill in the blanks below: *[3 points]*

$$A = \begin{pmatrix} -\frac{1}{3} & -2/3 & -\frac{2}{3} \\ \frac{2}{3} & -2/3 & \frac{1}{3} \\ \frac{2}{3} & 1/3 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1/3 & 2/3 & 2/3 \\ -2/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & -2/3 \end{pmatrix}.$$

(Recall) Λ has eigenvalues on its diagonal

Q has (normalized, to unit norm) eigenvectors as columns.

We have $\lambda_2 = 0$ since $\lambda_1\lambda_2\lambda_3 = 0$ and $\lambda_1, \lambda_3 \neq 0$

Hence $\Lambda = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Also, $(A - \lambda_2 I)\vec{q}_2 = \vec{0}$, or, $A\vec{q}_2 = \vec{0}$

Using elimination, $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

(back-substitution yields) $\vec{q}_2 = \alpha \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ (or find $\vec{q}_2 \in N(A)$)

choose α s.t $\|\vec{q}_2\| = 1$

We get $\vec{q}_2 = \frac{1}{3}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$