NAME: SOLUTIONS Quiz 11, MTH314-001

1. Answer the following questions about $A \in \mathbb{R}^{3\times 3}$ below.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of A. *[4 points]*

$$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)(-\lambda)(2-\lambda)$$

Setting $|A-\lambda I| = 0$, we get $(1-\lambda)(-\lambda)(2-\lambda) = 0$

$\Rightarrow \lambda = 0$ or $\lambda = 1$ or $\lambda = 2$

(b) Calculate the eigenvectors of A for the two smallest eigenvalues. *[6 points]*

$\boxed{\lambda = 0}$ When $\lambda = 0$, $A - \lambda I = A$. We solve $A\vec{x} = \vec{0}$ (nullspace of A)

Using elimination,

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_2]{} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(reduced row echelon form)

We compute the special solution (free var. is x_3) $\vec{x} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ this is the eigenvector corresponding to $\lambda = 0$ (note: any scalar multiple of $\vec{x}$ is also an eigenvector)

$\boxed{\lambda=1}$ Here, $A-\lambda I = A-I = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

to find the corresponding eigenvector, we solve $(A-\lambda I)\vec{y}=\vec{0}$

(find $\vec{y} \in N(A-\lambda I)$)

Using elimination,

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_2]{} \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(upper triangular)

we could solve for $\vec{y}$ here by back-substitution

$-x_2 + x_3 = 0$ — ⓐ

$-x_2 = 0$ — ⓑ

$x_3 = 0$ — ⓒ

from ⓑ and ⓒ, we have $x_2 = x_3 = 0$.

x_1 is a free var. choose $x_1 = 1$

$\Rightarrow \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\xrightarrow[]{R_1 \to R_1 - R_3} \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[]{R_1 \to R_1 - R_2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2/(-1)]{} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(reduced row echelon form)

we now find a special solution

(x_1 is the free var)

$\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the eigenvector corresponding to $\lambda=1$

(note: any scalar multiple of $\vec{y}$ is also an eigenvector)