CHAPTER 3

① For the matrices below, find a basis for the rowspace, a basis for the column space, a basis for the nullspace, and a basis for the left nullspace.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

What is the dimension of each of the fundamental subspaces?

② Find every possible solution, $\vec{x}$, to the system below

$$\begin{pmatrix} -1 & 3 & 1 & 2 \\ 2 & -6 & 4 & 8 \\ 0 & 0 & -2 & -4 \end{pmatrix} \vec{x} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$$

③ Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

① For the matrices below, find a basis for the rowspace, a basis for the column space, a basis for the nullspace, and a basis for the left nullspace.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

(Answer sketches)

(a)

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 2 & 1 & 4 & b_2 \\ 4 & 7 & 8 & b_3 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1}]{} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & -5 & 0 & b_2 - 2b_1 \\ 0 & -5 & 0 & b_3 - 4b_1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & -5 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

Note: augmented (generic) right hand side allows me to write basis for $N(A^T)$ not necessary for other subspaces

$$\xrightarrow[R_2 \to R_2/(-5)]{} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & 1 & 0 & \frac{-b_2 + 2b_1}{5} \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right] \xrightarrow[]{R_1 \to R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -\frac{1}{5}b_1 + \frac{3}{5}b_2 \\ 0 & 1 & 0 & -b_2/5 + \frac{2}{5}b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

← pivot rows

pivot cols

free col

Basis for row space $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Basis for nullspace $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

Basis for column space $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\}$

Basis for left nullspace $\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

① For the matrices below, find a basis for the rowspace, a basis for the column space, a basis for the nullspace, and a basis for the left nullspace.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

(Answer sketches)

(b)

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & b_1 \\ 2 & 1 & 3 & 2 & b_2 \\ 3 & 4 & 5 & 6 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & b_1 \\ 0 & -5 & 7 & 0 & b_2 - 2b_1 \\ 0 & -5 & 11 & 3 & b_3 - 3b_1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & b_1 \\ 0 & -5 & 7 & 0 & b_2 - 2b_1 \\ 0 & 0 & 4 & 3 & b_3 - b_2 - b_1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3/4]{R_2 \to R_2/(-5)} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & b_1 \\ 0 & 1 & -7/5 & 0 & \frac{2}{5}b_1 - \frac{1}{5}b_2 \\ 0 & 0 & 1 & 3/4 & -\frac{1}{4}b_1 - \frac{1}{4}b_2 + \frac{1}{4}b_3 \end{array}\right] \xrightarrow[R_2 \to R_2 + \frac{7}{5}R_3]{R_1 \to R_1 + 2R_3} \left[\begin{array}{cccc|c} 1 & 3 & 0 & \frac{5}{2} & \sim \\ 0 & 1 & 0 & \frac{21}{20} & \sim \\ 0 & 0 & 1 & \frac{3}{4} & \sim \end{array}\right] \xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{13}{20} & \sim \\ 0 & 1 & 0 & \frac{21}{20} & \sim \\ 0 & 0 & 1 & \frac{3}{4} & \sim \end{array}\right]$$

pivot cols; free col

Basis for $N(A^T) = \{\vec{0}\}$ (Why? all 3 rows are pivot rows; no combination of rows gives $\vec{0}$ other than the trivial (zero) combination)

Basis for $C(A^T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -13/20 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 21/20 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3/4 \end{pmatrix} \right\}$

Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\}$

Basis for $N(A) = \left\{ \begin{pmatrix} 13/20 \\ -21/20 \\ -3/4 \\ 1 \end{pmatrix} \right\}$

#2 Performing elimination

$$\left[\begin{array}{cccc|c} -1 & 3 & 1 & 2 & 4 \\ 2 & -6 & 4 & 8 & 4 \\ 0 & 0 & -2 & -4 & -4 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cccc|c} -1 & 3 & 1 & 2 & 4 \\ 0 & 0 & 6 & 12 & 12 \\ 0 & 0 & -2 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + \frac{1}{3}R_2} \left[\begin{array}{cccc|c} -1 & 3 & 1 & 2 & 4 \\ 0 & 0 & 6 & 12 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

upper triangular

$$\xrightarrow{R_2 \to R_2/6} \left[\begin{array}{cccc|c} -1 & 3 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} -1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1/(-1)} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

reduced row echelon form

pivot cols.

Note: rank $(A) = r = 2$ (# pivot cols)

Column space

not asked in problem; written here for illustration

$$C(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \right\}$$

$$\text{Basis for } C(A) = \left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \right\}$$

$$\dim C(A) = 2$$

Null space
(sol'ns to $A\vec{x} = \vec{0}$)

2 free cols $\Rightarrow$ 2 free vars. (2 special sol^{n}s)

free vars

$$\left.\begin{aligned} x_1 - 3x_2 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}\right\}$$

$$s_1 = \begin{pmatrix} 3 \\ \boxed{1} \\ 0 \\ \boxed{0} \end{pmatrix} \quad (\text{obtained by setting } x_2 = 1, x_4 = 0)$$

$$s_2 = \begin{pmatrix} 0 \\ \boxed{0} \\ -2 \\ \boxed{1} \end{pmatrix} \quad (\text{— — } x_2 = 0, x_4 = 1)$$

$$N(A) = \text{span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$\vec{x}_n = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \quad \text{where } x_2, x_4 \in \mathbb{R}$$

All solutions to $A\vec{x} = \vec{b}$

$$\vec{x} = \underbrace{\vec{x}_p}_{\text{particular sol}^n} + \underbrace{\vec{x}_n}_{\text{nullspace sol}^n}$$

$$\vec{x}_p = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad (\text{obtained by setting } x_2 = x_4 = 0)$$

$$\left\{\begin{aligned} x_1 - 3x_2 &= -2 \\ x_3 + 2x_4 &= 2 \end{aligned}\right.$$

③ Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(a) Let $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{bmatrix}$ Performing elimination,

$$\begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 2 & 3 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \underline{0} \end{bmatrix}}_{\text{upper triangular}}$$

$A \in \mathbb{R}^{3\times3}$ has rank 2; hence the set of vectors is <u>not</u> linearly independent

(b) $(1,1,3)$ and $(0,2,1)$ do not lie on the same line; hence they are linearly independent

alternatively, let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{\text{pivot cols}}$$

no. of columns = rank = r
(linearly independent cols)

CHAPTER 4

① Given $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -3 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}$

(a) find the projection of $\vec{b}$ onto $C(A)$

(b) Find the error vector $\vec{e} = \vec{b} - P\vec{b}$. Check that $\vec{e}$ is perpendicular to $C(A)$.

② Answer the questions about $A \in \mathbb{R}^{2\times 2}$ below.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

(a) Write a basis for $N(A)$

(b) Write a basis for $C(A^T)$

(c) Find the projection of $(1,1) \in \mathbb{R}^2$ onto $N(A)$.

(d) Find the projection of $(1,1) \in \mathbb{R}^2$ onto $C(A^T)$.

(e) Write the projection matrix $P \in \mathbb{R}^{2\times 2}$ that projects vectors onto $N(A)$.

③ Find the best least squares fit by

(a) a linear function (b) a quadratic function to the data

x	-1	0	1	2
y	0	1	3	9

④ Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$.

(a) Use the Gram-Schmidt process to find an orthonormal basis for $C(A)$.

(b) Find the QR factorization of A.

(c) Solve the least-squares problem $A\vec{x} = \vec{b}$.

① Given $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -3 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}$

(a) find the projection of $\vec{b}$ onto $C(A)$

(b) Find the error vector $\vec{e} = \vec{b} - P\vec{b}$. Check that $\vec{e}$ is perpendicular to $C(A)$.

(a) We first note that $3(\text{col } 1) + (\text{col } 2) = \vec{0}$ for matrix A (dependent columns).

Hence $C(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Let $B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Then $B^T B = 2$, $(B^T B)^{-1} = \frac{1}{2}$ and $B^T \vec{b} = 2$.

Hence projection of $\vec{b}$ onto $C(A)$ is $\vec{p} = \underbrace{B(B^T B)^{-1} B^T}_{P} \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(b) $\vec{e} = \vec{b} - P\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

all vectors in $C(A)$ are linear multiples of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Since $(-1, 1, 1) \cdot (1, 0, 1) = 0$, $\vec{e}$ is perpendicular to $C(A)$.

② Answer the questions about $A \in \mathbb{R}^{2\times 2}$ below.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$$

(a) Write a basis for $N(A)$

(b) Write a basis for $C(A^T)$

(c) Find the projection of $(1,1) \in \mathbb{R}^2$ onto $N(A)$.

(d) Find the projection of $(1,1) \in \mathbb{R}^2$ onto $C(A^T)$.

(e) Write the projection matrix $P \in \mathbb{R}^{2\times 2}$ that projects vectors onto $N(A)$.

Performing elimination on A,

$$\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \leftarrow \text{pivot row}$$

pivot col

Basis for $N(A) = \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$

Basis for $C(A^T) = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$

Let $\vec{b} = (1,1)$

Let $A_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. Then $A_1^T A_1 = 10$, $(A_1^T A_1)^{-1} = \frac{1}{10}$, $A_1^T \vec{b} = -2$

$$\vec{p}_1 = P_{N(A)} \vec{b} = A_1 (A_1^T A_1)^{-1} A_1^T \vec{b} = \left(-\tfrac{1}{5}\right) \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -1/5 \end{pmatrix}$$

} Projection onto $N(A)$

Let $A_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Then $A_2^T A_2 = 10$, $(A_2^T A_2)^{-1} = \frac{1}{10}$, $A_2^T \vec{b} = 4$

$$\vec{p}_2 = P_{C(A^T)} \vec{b} = A_2 (A_2^T A_2)^{-1} A_2^T \vec{b} = \frac{2}{5} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 6/5 \end{pmatrix}$$

} Projection onto $C(A^T)$

$$P_{N(A)} = A_1 (A_1^T A_1)^{-1} A_1^T = \frac{1}{10} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}$$

} Projection matrix onto $N(A)$

③ Find the best least squares fit by
(a) a linear function (b) a quadratic function to the data

x	-1	0	1	2
y	0	1	3	9

(a) Linear function has form $ax + b = y$

$$\begin{aligned} a(-1) + b &= 0 \\ a(0) + b &= 1 \\ a(1) + b &= 3 \\ a(2) + b &= 9 \end{aligned} \quad \rightsquigarrow \quad \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_{\vec{y}}$$

Least-Squares solution

$$A^T A \vec{x} = A^T \vec{y}$$

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad A^T \vec{y} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$$

$$A^T A \vec{x} = A^T \vec{y} \iff \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 21 \\ 13 \end{bmatrix}$$

using elimination

$$\left[\begin{array}{cc|c} 6 & 2 & 21 \\ 2 & 4 & 13 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 6 & 2 & 21 \\ 0 & 10/3 & 6 \end{array}\right]$$

back substitution yields $b = 9/5$ and $a = \frac{29}{10}$

③ Find the best least squares fit by
(a) a linear function (b) a quadratic function to the data

x	-1	0	1	2
y	0	1	3	9

(b) We want to fit a quadratic function ax^2+bx+c to the given data.

we have
$$a(-1)^2 + b(-1) + c = 0$$
$$a(0)^2 + b(0) + c = 1$$
$$a(1)^2 + b(1) + c = 3$$
$$a(2)^2 + b(2) + c = 9$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_{\vec{y}}$$

want to solve $A^TA\vec{x} = A^T\vec{y}$

following same procedure as in part (a), $A^TA = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}$ and $A^T\vec{y} = \begin{bmatrix} 39 \\ 21 \\ 13 \end{bmatrix}$

solving these equations yields

$$a = \frac{5}{4}, \quad b = \frac{33}{20} \quad \text{and} \quad c = \frac{11}{20}$$

#4

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

Projection of $\vec{b}$ onto $C(A)$ $\qquad \vec{p} = A(A^TA)^{-1}A^T\vec{b}$

$$A^TA = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} \qquad (A^TA)^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 66 \\ 36 \end{bmatrix} = 6\begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

$$\vec{p} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix} \right) \left(6 \begin{bmatrix} 11 \\ 6 \end{bmatrix} \right)$$

not asked in problem; provided here for illustration

$$= 3 \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} = \begin{pmatrix} 15 \\ 6 \\ 15 \end{pmatrix}$$

$$A = \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \\ \left(\begin{matrix} 2 \\ 1 \\ 2 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right) \end{array}$$

Gram Schmidt

Step 1 $\quad \|\vec{a}_1\| = 3 \quad$ and $\quad \vec{q}_1 = \dfrac{\vec{a}_1}{\|\vec{a}_1\|} = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$

$$\vec{a}_2 \cdot \vec{q}_1 = \frac{5}{3}$$

Step 2 $\quad \vec{b}_2 = \vec{a}_2 - (\vec{a}_2 \cdot \vec{q}_1)\,\vec{q}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{3}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 2/3 \end{pmatrix} = \begin{pmatrix} -1/9 \\ 4/9 \\ -1/9 \end{pmatrix}$

$$\|\vec{b}_2\| = \sqrt{2}/3$$

Hence $\quad \vec{q}_2 = \dfrac{\vec{b}_2}{\|\vec{b}_2\|} = \dfrac{-1}{3\sqrt{2}}\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$

QR factorization $\quad A = QR = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{pmatrix}$

Least Squares solution is

$$A^T A \vec{x} = A^T \vec{b}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

Using QR factorization, $A = QR$

$$(QR)^T (QR) \vec{x} = (QR)^T \vec{b}$$

$$R^T \underbrace{(Q^T Q)}_{I} R \vec{x} = R^T Q^T \vec{b}$$

$$R\vec{x} = Q^T \vec{b}$$

$$R = \begin{pmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{pmatrix} \quad Q = \begin{pmatrix} 2/3 & -1/3\sqrt{2} \\ 1/3 & 4/3\sqrt{2} \\ 2/3 & -1/3\sqrt{2} \end{pmatrix}$$

Hence $Q^T \vec{b} = \begin{pmatrix} 22 \\ -\sqrt{2} \end{pmatrix}$

Solving $R\vec{x} = Q^T \vec{b}$ yields $\vec{x} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$

CHAPTER 5

① Answer the following questions about $A \in \mathbb{R}^{3\times 3}$ below

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & 3 \\ 3 & 10 & 3 \end{bmatrix}$$

(a) Calculate the determinant of A by reducing it to upper triangular form

(b) Calculate $|A^{-1}| = \det(A^{-1})$

(c) Calculate $|-A| = \det(-A)$

(d) Calculate the determinant of the matrix below

$$\begin{bmatrix} 1 & 0 & 3 & 3 \\ 2 & 0 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 10 & 3 \end{bmatrix}$$

note: see Quiz #10 for work|strategy

solutions (a) $|A| = 27$ (b) $1/27$

(c) $|-A| = -27$ (d) -54