Name: SOLUTIONS

READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything except pens, pencils and erasers.
- If you need scratch paper, use the back of the previous page.
- Without fully opening the exam, check that you have pages 1 through 13.
- Fill in your name, etc. on this first page.
- **Show all your work.** Write your answers clearly! Include enough steps for the grader to be able to follow your work. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- There is no talking allowed during the exam.
- You will be given exactly 50 minutes for this exam.

I have read and understand the above instructions: ______________________

SIGNATURE

Quick Answer Questions. No partial credit available.

1. Fill in the blanks below. No justification necessary.

(a) (2 points) The cosine of the angle between the vectors $(-1, 1, 0)$ and $(2, 1, -1)$ is $-\frac{1}{2\sqrt{3}}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{(-1)(2)+(1)(1)+(0)(-1)}{\sqrt{(-1)^2+1^2+0}\sqrt{2^2+1^2+(-1)^2}} = \frac{-1}{\sqrt{2}\sqrt{6}} = \frac{-1}{2\sqrt{3}}$$

(b) (2 points) Suppose $A, B \in \mathbb{R}^{n\times n}$, with A invertible. Then $\left(A^{-1} + BA^{\mathrm{T}}\right)^{\mathrm{T}} =$ $(A^{-1})^T + AB^T$
(Use properties of transposes to simplify the given expression)

$$(A^{-1}+BA^T)^T = (A^{-1})^T + (BA^T)^T$$
$$= (A^{-1})^T + AB^T$$
$$= (A^T)^{-1} + AB^T$$

(c) (2 points) Every matrix has an LU decomposition after we reorder its Rows.

(d) (2 points) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $(AB)^{-1} =$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad (AB)^{-1} = (AB)^T$$

Since AB (and A, B) are permutation matrices

(e) (2 points) True or False: For $M, N \in \mathbb{R}^{n\times n}$, it is always true that $(MP)^2 = M^2P^2$. **T** [✓ **F**]

$(MP)^2 = MPMP \neq MMPP$ Since $MP \neq PM$ In general

Counterexample: $M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $MP = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ but $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $M^2P^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $(MP)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $P^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. Determine whether the following sets form subspaces of $\mathbb{R}^3$. Answer yes or no.

   (a) (2 points) $\{(x_1, x_2, x_3) \mid x_1 + x_3 = 1\}$

   NO

$\vec{a} = (1,0,0)$

$\vec{b} = (0,0,1)$

elements from set

$\vec{a}+\vec{b} = (1,0,1)$

$x_1 + x_3 = 2 \neq 1$

No closure under addition

   (b) (2 points) $\{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$

   YES

Elements from set

$\vec{a} = (x_1, x_1, x_1)$

$\vec{b} = (y_1, y_1, y_1)$

$\vec{a}+\vec{b} = (x_1+y_1, x_1+y_1, x_1+y_1)$

$c\vec{a} = (cx_1, cx_1, cx_1)$

(closure under both addition, scalar multiplication)

3. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$. Answer yes or no.

   (a) (2 points) The set of all 2×2 lower triangular matrices.

   YES

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} = \begin{pmatrix} a+p & 0 \\ b+q & c+r \end{pmatrix} \rightarrow \text{lower triangular}$$

$$\lambda \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} \lambda a & 0 \\ \lambda b & \lambda c \end{pmatrix} \rightarrow \text{lower triangular}$$

(closure under addition, scalar multiplication)

   (b) (2 points) The set of all 2×2 matrices A such that $a_{12} = 1$.

   NO

$$\vec{p} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \quad \vec{q} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\vec{p}+\vec{q} = \begin{pmatrix} 1 & \boxed{2} \\ 0 & 3 \end{pmatrix} \neq 1$$

no closure under addition

Complete Explanation Questions. Provide complete justifications for your responses.

4. (20 points) **Matrix Factorization:** Compute the $PLDU$ factorization of the matrix $A \in \mathbb{R}^{3\times 3}$ below. That is, find a permutation matrix $P \in \mathbb{R}^{3\times 3}$, a lower triangular matrix $L \in \mathbb{R}^{3\times 3}$, a diagonal matrix $D \in \mathbb{R}^{3\times 3}$, and an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ with ones on the diagonal, such that $A = PLDU$. Make sure you write down your steps methodically.

$$A = \begin{pmatrix} 0 & -6 & -3 \\ -3 & 5 & 1 \\ 6 & -16 & 3 \end{pmatrix}.$$

Let's perform elimination

Swap rows① and ②, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $PA = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -6 & -3 \\ 6 & -16 & 3 \end{bmatrix}$.

row③ → row③ + 2row①, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $E_{31}PA = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -6 & -3 \\ 0 & -6 & 5 \end{bmatrix}$.

row③ → row③ - row②, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $E_{32}E_{31}PA = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -6 & -3 \\ 0 & 0 & 8 \end{bmatrix}$.

dividing by the pivots,
$$E_{32}E_{31}PA = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & -5/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now use $L = (E_{32}E_{31})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$ and $P^{-1} = P^T = P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

then
$$A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -5/3 & -1/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Additional Work Space:

5. **Inverses:** Solve

$$\begin{aligned} x + 3y + 3z &= 2 \\ x + 4y + 3z &= 1 \\ 2x + 7y + 7z &= 1 \end{aligned}$$

for $(x, y, z) \in \mathbb{R}^3$ using the two steps below. Remember to show your work!

(a) (16 points) Find the inverse of the matrix associated with the system of equations.

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \text{row}② \rightarrow \text{row}② - \text{row}① \\ \text{row}③ \rightarrow \text{row}③ - 2\text{row}① \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad \text{row}③ \rightarrow \text{row}③ - \text{row}②$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad \text{row}① \rightarrow \text{row}① - 3\text{row}③$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \underbrace{-1 \quad -1 \quad 1}_{A^{-1}} \end{array}\right] \quad \text{row}① \rightarrow \text{row}① - 3\text{row}②$$

$$\boxed{A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}$$

(b) (4 points) Use the inverse from above to solve the system of equations.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ -2 \end{bmatrix}$$

$$(x, y, z) = (11, -1, -2)$$

Additional Work Space:

6. **Vectors and Properties**

   (a) (8 points) Find non-zero unit vectors $\vec{u}, \vec{v}$ and $\vec{w}$ that are perpendicular to $(1, 0, 1, 0)$ and to each other. Show all your work.

Let $\vec{a} = (1, 0, 1, 0)$

Choose $\vec{u} = (0, 1, 0, 0)$. We have $\|\vec{v}\| = 1$ and $\vec{u} \cdot \vec{a} = 0$.

$\vec{v} = (0, 0, 0, 1)$. Then $\|\vec{v}\| = 1$ and $\vec{v} \cdot \vec{a} = \vec{v} \cdot \vec{u} = 0$

$\vec{w} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0\right)$. Then $\|\vec{w}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$

and $\vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = \vec{a} \cdot \vec{w} = 0$.

$\Rightarrow \vec{u}, \vec{v}, \vec{w}$ are perpendicular to $(1, 0, 1, 0)$ and each other.

   (b) (4 points) For the vectors $\vec{u}, \vec{v}, \vec{w}$ from part (a), find the angle θ between $\vec{u}$ and $(3\vec{v} + \sqrt{2}\vec{w})$.

We have $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = 0$

Hence $\vec{u} \cdot (3\vec{v} + \sqrt{2}\,\vec{w}) = 3\,\vec{u} \cdot \vec{v} + \sqrt{2}\,\vec{u} \cdot \vec{w} = 0$

$\Rightarrow \vec{u}$ is perpendicular to $(3\vec{v} + \sqrt{2}\,\vec{w})$.

(c) (6 points) Let $\vec{p}, \vec{q} \in \mathbb{R}^2$. When is $\|\vec{p}+\vec{q}\| = \|\vec{p}-\vec{q}\|$? Justify your answer. (Do not provide a specific example; find a general condition on $\vec{p}, \vec{q}$.)

Since $\|\vec{p}+\vec{q}\| = \|\vec{p}-\vec{q}\|$, we have

$$\|\vec{p}+\vec{q}\|^2 = \|\vec{p}-\vec{q}\|^2$$

$$\Rightarrow (\vec{p}+\vec{q})\cdot(\vec{p}+\vec{q}) = (\vec{p}-\vec{q})\cdot(\vec{p}-\vec{q})$$

$$\Rightarrow \vec{p}\cdot\vec{p} + \vec{q}\cdot\vec{p} + \vec{p}\cdot\vec{q} + \vec{q}\cdot\vec{q} = \vec{p}\cdot\vec{p} - \vec{q}\cdot\vec{p} - \vec{p}\cdot\vec{q} + \vec{q}\cdot\vec{q}$$

$$\Rightarrow 2\vec{q}\cdot\vec{p} = -2\vec{q}\cdot\vec{p} \quad \text{(commutative law)}$$

$$\Rightarrow 4\vec{q}\cdot\vec{p} = 0$$

$\Rightarrow \vec{p}, \vec{q}$ are perpendicular.

7. **Vector Spaces and Subspaces**

(a) (8 points) Let V be the set of all ordered pairs of real numbers (i.e., vectors in $\mathbb{R}^2$) with addition defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$

and scalar multiplication defined by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2), \qquad \text{where} \quad \alpha \in \mathbb{R}.$$

Vector addition for this system is defined in an unusual way, and consequently we use the symbol $\oplus$ to avoid confusion with ordinary addition. Is V a vector space with these operations? Justify your answer – if it is a vector space, show briefly that the axioms are satisfied; if it is not a vector space, provide a counterexample for an axiom that is not satisfied.

No. V is not a vector space for these operations.

Counterexample Choose $\vec{a} = (1, 2)$, $\vec{b} = (3, 1)$. Note that $\vec{a}, \vec{b} \in V$.

Let's check axiom ①

$$\vec{a} \oplus \vec{b} = \vec{b} \oplus \vec{a}$$

$\vec{a} \oplus \vec{b} = (-2, 1)$, but $\vec{b} \oplus \vec{a} = (2, -1)$.

Clearly $\vec{a} \oplus \vec{b} \neq \vec{b} \oplus \vec{a}$.

Axiom ① not satisfied. Hence V is not a vector space for these operations.

(b) (16 points) For what values of a does the following linear system of equations have a unique solution?

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 1 \\ -x & + & y & + & 3z & = & 2 \\ x & - & y & + & az & = & 3. \end{array}$$

Next, determine whether the vector $(1, 2, 3)$ is in the span of the set of vectors

$$\{(1, -1, 1),\ (2, 1, -1),\ (1, 3, 2)\}.$$

If $(1, 2, 3)$ is in the span, express it as a linear combination of the other three vectors. Remember to show your work!

Perform elimination on $[A \vdots \vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 1 & 3 & 2 \\ 1 & -1 & a & 3 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & -3 & a-1 & 2 \end{array}\right]$ row ② → row② + row ①
row ③ → row③ − row ①

$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 0 & a+3 & 5 \end{array}\right]$ row③ → row③ + row ②

If $\boxed{a \neq -3}$, we have a full set of pivots and hence a unique solution.

$\boxed{\text{Yes, } (1,2,3) \in \text{span}\{(1,-1,1),\ (2,1,-1),\ (1,3,2)\}}$

Why? $\text{span}\{(1,-1,1),\ (2,1,-1),\ (1,3,2)\} = C(A)$

Since $a \neq -3$, $(1,2,3) \in C(A)$. We find the specific linear combination by back-substitution.

Back substitution gives $z = 1$, $y = -\frac{1}{3}$ and $x = \frac{2}{3}$.

Hence $\boxed{(1,2,3) = \frac{2}{3}(1,-1,1) - \frac{1}{3}(2,1,-1) + (1,3,2)}$

Additional Work Space:

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	10	
3	8	
4	20	
6	20	
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