Name: SOLUTIONS

READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything except pens, pencils and erasers.
- If you need scratch paper, use the back of the previous page.
- Without fully opening the exam, check that you have pages 1 through 13.
- Fill in your name, etc. on this first page.
- **Show all your work.** Write your answers clearly! Include enough steps for the grader to be able to follow your work. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- There is no talking allowed during the exam.
- You will be given exactly 50 minutes for this exam.

I have read and understand the above instructions: ______________________

SIGNATURE

Quick Answer Questions. No partial credit available; No justification necessary.

1. Fill in the blanks below.

(a) (2 points) Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$ and $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$.

Then $P^3 = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$.

note: P is a projection matrix hence $P^2 = P$
Therefore, $P^3 = P(P^2) = PP = P^2 = P$

(b) (2 points) Let A be an $n \times n$ matrix.

(i) If the row space of A is $\mathbb{R}^n$, then the column space of A is $\mathbb{R}^n$

(ii) If the left nullspace of A is $\mathbb{R}^n$, then the column space of A is $\{\vec{0}\}$

note: $A \in \mathbb{R}^{n\times n}$
$C(A^T) = \mathbb{R}^n$
$\Rightarrow$ rank $r = n$
$\Rightarrow C(A) = \mathbb{R}^n$
$N(A^T)$ and $C(A)$ are orthogonal subspace

(c) (1 point) The determinant of $\begin{pmatrix} 5 & 2 & 7 \\ 0 & 2 & c \\ 0 & 2 & 3 \end{pmatrix}$ is $10(3-c) = 30-10c$.

For what value(s) of c is the above matrix singular? $c = 3$.

note: $\begin{pmatrix} 5 & 2 & 7 \\ 0 & 2 & c \\ 0 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 2 & 7 \\ 0 & 2 & c \\ 0 & 0 & 3-c \end{pmatrix}$. upper triangular
singular $\Rightarrow \det(A) = 0$
hence $10(3-c) = 0$
$\Rightarrow c = 3$

(d) (2 points) Suppose

pivot cols

$$A = \begin{bmatrix} 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{pivot rows} \\ \leftarrow \\ \end{array}$$

here $A \in \mathbb{R}^{3\times 5}$

The row space of A has dimension = 2.

The column space of A has dimension = 2.

The null space of A^{T} has dimension = 1.

$N(A^{\mathrm{T}})^{\perp} = \{\vec{y} \mid \vec{y} \perp N(A^{\mathrm{T}})\} = C(A) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

note:
$\dim C(A) = $ rank $=$ # of pivot cols
$\dim C(A^T) = \dim C(A)$
$\dim N(A^T) = m - r = 3 - 2 = 1$
$N(A^T)$ and $C(A)$ are orthogonal subspaces

(e) (1 point) True or False: Let A and B be any two matrices so that the product AB is defined. Then $\text{rank}(A) \leq \text{rank}(AB)$. rank (AB) $\leqslant$ rank (A) T F

(f) (2 points) Construct a matrix A with the following property or say why it is impossible:

$A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has a solution and $A^{\mathrm{T}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. IMPOSSIBLE

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in C(A)$ and $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \in N(A^T)$

every vector in $N(A^T)$ should be perpendicular to every vector in $C(A)$.

However $(1, 2, 3) \cdot (-2, 1, 1) = 3 \neq 0$

Additional Work Space:

Complete Explanation Questions. Provide complete justifications for your responses.

2. Suppose

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 4 & 6 \\ 7 & -1 & 5 & 14 & 36 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 22 \end{bmatrix}.$$

(a) (8 points) Use elimination to find the reduced row echelon form of A.

$$[A \vdots \vec{b}] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 5 & 3 \\ 1 & 1 & 3 & 4 & 6 & 4 \\ 7 & -1 & 5 & 14 & 36 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{array}\right]$$

(upper triangular form)

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3/2]{} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] = R$$

(reduced row echelon form)

pivot cols

(b) (6 points) Find a basis for the column space of A.

Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} \right\}$

note: col^s 1, 2 and 4 are the pivot col^s.

(c) (8 points) Find a basis for the nullspace of A.

Free columns are col 3 and col 5.

Solving $A\vec{x} = \vec{0}$ (or, equivalently $R\vec{x} = \vec{0}$) gives

$$\left.\begin{aligned} x_1 \quad + x_3 \quad + 3x_5 &= 0 \\ x_2 + 2x_3 \quad - x_5 &= 0 \\ x_4 + x_5 &= 0 \end{aligned}\right\}$$

where x_3 and x_5 are the free variables

special solutions

(setting $x_3 = 1$, $x_5 = 0$) $\vec{s}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(setting $x_3 = 0$, $x_5 = 1$) $\vec{s}_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Basis for $N(A) = \left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

(d) (8 points) Give the complete solution to $A\vec{x} = \vec{b}$.

Complete solution is $\vec{x} = \underbrace{\vec{x}_p}_{\text{particular sol}^n} + \underbrace{\vec{x}_n}_{\text{nullspace sol}^n}$ $\quad \vec{x}_n = x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

where $x_3, x_5 \in \mathbb{R}$.

Particular solution

$$\left.\begin{aligned} x_1 \quad + x_3 \quad + 3x_5 &= 1 \\ x_2 + 2x_3 \quad - x_5 &= -1 \\ x_4 + x_5 &= 1 \end{aligned}\right\}$$

(setting $x_3 = x_5 = 0$) $\vec{x}_p = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

pivot vars: x_1, x_2, x_4

free vars: x_3, x_5

Complete sol^n $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Additional Work Space:

3. Given

$$A = \begin{bmatrix} 3 & -6 & 9 \\ 4 & -8 & 12 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} -1 \\ 7 \\ 2 \\ 3 \end{bmatrix},$$

(a) (6 points) Find the projection of $\vec{b}$ onto $N(A^{\mathrm{T}})$.
(*Hint: The left nullspace of A contains vectors of the form $\vec{y} = (\_,\_,0,0)$. Can you quickly find the missing entries by inspection?*)

We note that $A \in \mathbb{R}^{4\times 3}$

$\dim N(A^T) = 4 - r$ where $r = \text{rank}(A) = \#$ pivot/independent columns.

From inspection, the columns of A are linearly independent.

Hence $\dim N(A^T) = 4-3 = 1$.

Using the provided hint (or by inspecting A), $N(A^T) = \text{span}\left\{ \begin{pmatrix} -4/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

Therefore, consider $B = \begin{pmatrix} -4/3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Let $\vec{p} = P_{N(A^T)} \vec{b} = B (B^T B)^{-1} B^T \vec{b}$

We have $B^T B = \frac{16}{9} + 1 = \frac{25}{9}$; $(B^T B)^{-1} = \frac{9}{25}$

$B^T \vec{b} = \frac{4}{3} + 7 = \frac{25}{3}$

Therefore, $\vec{p} = \begin{pmatrix} -4/3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \left(\frac{9}{25}\right)\left(\frac{25}{3}\right) = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

(b) (15 points) If the Gram-Schmidt process is applied to determine an orthonormal basis for $C(A)$ and a QR factorization of A, then, after the first two orthonormal vectors $\vec{q}_1$ and $\vec{q}_2$ are computed, we have

$$Q = \begin{array}{c} \begin{pmatrix} \frac{3}{5} & 0 & \underline{0} \\ \frac{4}{5} & 0 & \underline{0} \\ 0 & 1 & \underline{0} \\ 0 & 0 & \underline{1} \end{pmatrix} \\ \vec{q}_1 \quad \vec{q}_2 \quad\quad \end{array}, \qquad R = \begin{pmatrix} 5 & -10 & \underline{15} \\ 0 & 1 & \underline{1} \\ 0 & 0 & \underline{1} \end{pmatrix}. \qquad \boxed{\text{Recall,} \quad A = \begin{bmatrix} 3 & -6 & 9 \\ 4 & -8 & 12 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\vec{q}_1 \quad \vec{q}_2 \quad \vec{q}_3$$

Finish the process. Determine $\vec{q}_3$ and fill in the third columns of Q and R.

We have

$$r_{13} = \vec{a}_3 \cdot \vec{q}_1 = \frac{27}{5} + \frac{48}{5} = \frac{75}{5} = 15$$

$$r_{23} = \vec{a}_3 \cdot \vec{q}_2 = 1$$

Let $\vec{v}_3 = \vec{a}_3 - (\vec{a}_3 \cdot \vec{q}_1)\,\vec{q}_1 - (\vec{a}_3 \cdot \vec{q}_2)\,\vec{q}_2$

$$= \begin{pmatrix} 9 \\ 12 \\ 1 \\ 1 \end{pmatrix} - 15 \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$r_{33} = \|\vec{v}_3\| = 1$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c) (10 points) Use the QR factorization to find the least squares solution of $A\vec{x} = \vec{b}$.

(Recall) $A = \begin{bmatrix} 3 & -6 & 9 \\ 4 & -8 & 12 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -1 \\ 7 \\ 2 \\ 3 \end{bmatrix}$ $Q = \begin{pmatrix} \frac{3}{5} & 0 & 0 \\ \frac{4}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 5 & -10 & 15 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Note: Least Squares $\Longleftrightarrow$ $A^T A \vec{x} = A^T \vec{b}$

using $A = QR$,

$$(QR)^T (QR)\vec{x} = (QR)^T \vec{b}$$

$$R^T \underbrace{(Q^T Q)}_{I} R\vec{x} = R^T Q^T \vec{b}$$

$$R\vec{x} = Q^T \vec{b}$$

We have $Q^T \vec{b} = \begin{pmatrix} 3/5 & 4/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3/5 + 28/5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$R\vec{x} = Q^T \vec{b}$ $\Longleftrightarrow$ $\begin{pmatrix} 5 & -10 & 15 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

backsubstitution yields

$$x_3 = 3$$

$$x_2 = 2 - 3 = -1$$

$$x_1 = \frac{5 + 10(-1) - 15(3)}{5} = -10$$

Hence least squares solution is $\vec{x} = \begin{pmatrix} -10 \\ -1 \\ 3 \end{pmatrix}$

4. Let

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & -2 & -3 \end{pmatrix}.$$

(a) (12 points) Calculate the determinant of A by reducing it to upper triangular form.

Performing elimination,

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & -2 & -3 \end{bmatrix}$$

$$P_{12} A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & -2 & -3 \end{bmatrix} \quad \text{(swapping rows (1) and (2))}$$

$$\xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & -4 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_3]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{(upper triangular form)}$$

Therefore,

$$\underbrace{|P_{12}|}_{\substack{-1 \\ \text{(one row swap)}}} |A| = (1)(1)(-3)(-1) = 3$$

$$\Longrightarrow |A| = -3$$

(b) (6 points) Calculate $|-A^{\mathrm{T}}| = \det(-A^{\mathrm{T}})$.

$$(Recall) \qquad A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & -2 & -3 \end{pmatrix} \qquad |A| = -3$$

Using $|A^T| = |A|$ and linearity of row operations,

$$|-A^T| = (-1)^4 |A^T| = |A| = -3$$

(c) (11 points) Evaluate

$$\underbrace{\begin{vmatrix} 0 & 1 & 2 & 3 \\ 2 & 2 & -1 & -1 \\ 1 & 1 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}_{B} + \underbrace{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 3 & 3 & 0 & 0 \\ 1 & 1 & -2 & -3 \end{vmatrix}}_{C}$$

We have $P_{2\leftrightarrow 3} P_{2\leftrightarrow 4} A = B$ (first swap $R_2 \leftrightarrow R_4$; then swap $R_2 \leftrightarrow R_3$)

Therefore $|P_{2\leftrightarrow 3}|\,|P_{2\leftrightarrow 4}|\,|A| = |B|$

$(-1)(-1)(-3) = |B|$

$\Rightarrow |B| = -3$

Similarly, $$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\det(\cdot)=1} A = C$$ (for matrix A, $R_1 \to R_1 + R_2$, $R_3 \to R_3 + R_2$)

$\Rightarrow |A| = |C| = -3$ (by linearity of row operations)

Therefore $|B| + |C| = -3 + (-3) = -6$

Additional Work Space:

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
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