Positive Definite Matrices

Defn A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called <u>positive definite</u> if all its eigenvalues are positive real numbers

note: We know the eigenvalues are real since $A = A^T$

- ⇝ positive semi-definite (eigenvalues are ≥ 0)
- ⇝ negative definite (— " — < 0)
- ⇝ negative semi-definite (— " — ≤ 0)

The following are equivalent Let $A = A^T$, $A \in \mathbb{R}^{n \times n}$

① $\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n$ with $\|\vec{x}\| > 0$

② all n pivots of A are positive ($A = LU$ ← upper triangular with pivots on diagonal)

lower triangular with 1's on diagonal

③ A is positive definite

RELATION BETWEEN PIVOTS AND EIGENVALUES

Let $A \in \mathbb{R}^{2\times 2}$ with $A^T = A$. Consider $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

First, note that $\lambda_1 \lambda_2 = |A| = ac - b^2$ and $\lambda_1 + \lambda_2 = \underbrace{a+c}_{\text{trace}(A)}$

Performing elimination on A

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \xrightarrow[R_2 \to R_2 - \frac{b}{a} R_1]{} \begin{pmatrix} a & b \\ 0 & c - \frac{b^2}{a} \end{pmatrix}$$

pivot 1

pivot 2

If $a>0$ and $c - \dfrac{b^2}{a} > 0$, then

$$\Rightarrow a+c > a + c - \frac{b^2}{a} > 0 \implies \lambda_1 + \lambda_2 > 0 \quad (\text{since } \lambda_1 + \lambda_2 = a+c)$$

$$\text{———} \ (*)$$

and

$$\Rightarrow a\left(c - \frac{b^2}{a}\right) = ac - b^2 = \det(A) > 0 \implies \lambda_1 \lambda_2 > 0 \quad (\text{since } |A| = \lambda_1 \lambda_2)$$

$$\text{———} \ (**)$$

from $(*)$ and $(**)$, we have λ_1 and $\lambda_2 > 0$

Example Is $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 2 & 5 \end{pmatrix}$ positive definite?

first note that $A = A^T$, so we reduce to upper triangular form and look at the signs of the pivots.

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 2 & 5 \end{pmatrix} \xrightarrow[R_3 \to R_3 + \frac{1}{3}R_1]{R_2 \to R_2 - \frac{1}{3}R_1} \begin{pmatrix} 3 & 1 & -1 \\ 0 & -1/3 & 7/3 \\ 0 & 7/3 & 14/3 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 7R_2} \begin{pmatrix} 3 & 1 & -1 \\ 0 & -1/3 & 7/3 \\ 0 & 0 & 35/3 \end{pmatrix}$$

pivot < 0

$\Rightarrow$ atleast one of the eigenvalues is ≤ 0.

Check: $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$ positive definite?

$\boxed{\text{Thm}}$ More generally, the number of positive eigenvalues of $A = A^T$ is equal to the number of positive pivots.